Sets

Sections 2.1 and 2.2 of Rosen
Spring 2018
CSCE 235H Introduction to Discrete Structures (Honors)
Course web-page: cse.unl.edu/~cse235h
Questions: Piazza
A set is a collection of objects.

For example:

- $S = \{s_1, s_2, s_3, \ldots, s_n\}$ is a finite set of $n$ elements
- $S = \{s_1, s_2, s_3, \ldots\}$ is a infinite set of elements.

$s_1 \in S$ denotes that the object $s_1$ is an element of the set $S$

$s_1 \notin S$ denotes that the object $s_1$ is not an element of the set $S$

LaTeX

- $S = \{s_1, s_2, s_3, \ldots, s_n\}$
- $s_i \in S$
- $si \notin S$
Sets of Numbers

- Using the package: `\usepackage{amssymb}`
  - Set of natural numbers: `$\mathbb{N}$`: may or may not include 0 (by default, it does)
  - Set of integer numbers: `$\mathbb{Z}$`
  - Set of rational numbers: `$\mathbb{Q}$`
  - Set of real numbers: `$\mathbb{R}$`
  - Set of complex numbers: `$\mathbb{C}$`
Outline

- Definitions: set, element
- Terminology and notation
  - Set equal, multi-set, bag, set builder, intension, extension, Venn Diagram (representation), empty set, singleton set, subset, proper subset, finite/infinite set, cardinality
- Proving equivalences
- Power set
- Tuples (ordered pair)
- Cartesian Product (a.k.a. Cross product), relation
- Quantifiers
- Set Operations (union, intersection, complement, difference), Disjoint sets
- Set equivalences (cheat sheet or Table 1, page 130)
  - Inclusion in both directions
  - Using membership tables
- Generalized Unions and Intersection
- Computer Representation of Sets
Introduction (1)

- We have already implicitly dealt with sets
  - Integers ($\mathbb{Z}$), rationals ($\mathbb{Q}$), naturals ($\mathbb{N}$), reals ($\mathbb{R}$), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an **unordered** collection of **unique** objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs
Introduction (2)

- **Definition**: The objects in a set are called **elements** or **members** of a set. A set is said to contain its elements.

- **Notation**, for a set $A$:
  - $x \in A$: $x$ is an element of $A$ \(\text{	extbackslash in}\)
  - $x \notin A$: $x$ is not an element of $A$ \(\text{	extbackslash notin}\)
Terminology (1)

• **Definition:** Two sets, A and B, are equal is they contain the same elements. We write $A = B$.

• Example:
  
  – $\{2,3,5,7\} = \{3,2,7,5\}$, because a set is unordered
  
  – Also, $\{2,3,5,7\} = \{2,2,3,5,3,7\}$ because a set contains unique elements
  
  – However, $\{2,3,5,7\} \neq \{2,3\}$
Terminology (2)

• A **multi-set** is a set where you specify the number of occurrences of each element: \( \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\} \) is a set where
  – \( m_1 \) occurs \( a_1 \) times
  – \( m_2 \) occurs \( a_2 \) times
  – …
  – \( m_r \) occurs \( a_r \) times

• In Databases, we distinguish
  – A set: elements cannot be repeated
  – A **bag**: elements can be repeated
Terminology (3)

• The **set-builder** notation
  \[ S = \{ x \mid (x \in \mathbb{Z}) \land (x=2k) \text{ for some } k \in \mathbb{Z} \} \]
  reads: \( S \) is the set that contains all \( x \) such that \( x \) is an integer and \( x \) is even

• A set is defined in **intension** when you give its set-builder notation
  \[ S = \{ x \mid (x \in \mathbb{Z}) \land (0 \leq x \leq 8) \land (x=2k) \text{ for some } k \in \mathbb{Z} \} \]

• A set is defined in **extension** when you enumerate all the elements:
  \[ S = \{0, 2, 4, 6, 8\} \]
Venn Diagram: Example

• A set can be represented graphically using a Venn Diagram
More Terminology and Notation (1)

• A set that has no elements is called the **empty set** or **null set** and is denoted \( \emptyset \).

• A set that has one element is called a **singleton set**.
  - For example: \{a\}, with brackets, is a singleton set
  - a, without brackets, is an element of the set \{a\}

• Note the subtlety in \( \emptyset \neq \{\emptyset\} \)
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set
More Terminology and Notation (2)

• **Definition**: A is said to be a *subset* of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B.

• That is, we have the equivalence:

$$A \subseteq B \iff \forall x \ (x \in A \Rightarrow x \in B)$$
More Terminology and Notation (3)

- **Theorem:** For any set $S$
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$

- The proof is in the book, an excellent example of a vacuous proof

*Theorem 1, page 120*
More Terminology and Notation (4)

• **Definition**: A set \( A \) that is a subset of a set \( B \) is called a **proper subset** if \( A \neq B \).

• That is there is an element \( x \in B \) such that \( x \notin A \).

• We write: \( A \subset B, A \subsetneq B \).

• In LaTeX: \( \subseteq, \subsetneq \).
More Terminology and Notation (5)

• Sets can be elements of other sets
• Examples
  – $S_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, c\}$
  – $S_2 = \{\{1\}, \{2, 4, 8\}, \{3\}, \{6\}, 4, 5, 6\}$
More Terminology and Notation (6)

• **Definition**: If there are exactly $n$ distinct elements in a set $S$, with $n$ a nonnegative integer, we say that:
  – $S$ is a *finite set*, and
  – The *cardinality* of $S$ is $n$. Notation: $|S| = n$.

• **Definition**: A set that is not finite is said to be *infinite*
More Terminology and Notation (7)

• Examples
  – Let $B = \{x \mid (x \leq 100) \land (x \text{ is prime})\}$, the cardinality of $B$ is $|B|=25$ because there are 25 primes less than or equal to 100.
  – The cardinality of the empty set is $|\emptyset|=0$
  – The sets $N$, $Z$, $Q$, $R$ are all infinite
Proving Equivalence (1)

• You may be asked to show that a set is
  – a subset of,
  – proper subset of, or
  – equal to another set.

• To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \iff \forall x(x \in A \implies x \in B)$
  – To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element $x$, $x \in A$ implies that $x$ is also in $B$.
  – Any proof method can be used.

• To prove that A is a proper subset of B, you must prove
  – A is a subset of B and
  – $\exists x \ (x \in B) \land (x \notin A)$
Proving Equivalence (2)

• Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  – $A \subseteq B$ and
  – $B \subseteq A$
• Logically speaking, you must show the following quantified statements:
  
  $$(\forall x (x \in A \Rightarrow x \in B)) \land (\forall x (x \in B \Rightarrow x \in A))$$

  we will see an example later..
Power Set (1)

- **Definition**: The power set of a set $S$, denoted $P(S)$, is the set of all subsets of $S$.

- **Examples**
  - Let $A = \{a, b, c\}$, $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$
  - Let $A = \{\{a, b\}, c\}$, $P(A) = \{\emptyset, \{\{a, b\}\}, \{c\}, \{\{a, b\}, c\}\}$

- **Note**: the empty set $\emptyset$ and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 120).
Power Set (2)

- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact**: Let $S$ be a set such that $|S| = n$, then
  $$|P(S)| = 2^n$$
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• Proving equivalences
• Power set
  • Tuples (ordered pair)
  • Cartesian Product (a.k.a. Cross product), relation
  • Quantifiers
• Set Operations (union, intersection, complement, difference), Disjoint sets
• Set equivalences (cheat sheet or Table 1, page 130)
  • Inclusion in both directions
  • Using membership tables
• Generalized Unions and Intersection
• Computer Representation of Sets
Tuples (1)

- Sometimes we need to consider **ordered** collections of objects
- **Definition:** The ordered n-tuple \((a_1, a_2, \ldots, a_n)\) is the ordered collection with the element \(a_i\) being the i-th element for \(i=1,2,\ldots,n\)
- Two ordered n-tuples \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) are equal iff for every \(i=1,2,\ldots,n\) we have \(a_i=b_i\) \((a_1, a_2, \ldots, a_n)\)
- A 2-tuple \((n=2)\) is called an **ordered pair**
Cartesian Product (1)

- **Definition**: Let $A$ and $B$ be two sets. The Cartesian product of $A$ and $B$, denoted $A \times B$, is the set of all ordered pairs $(a,b)$ where $a \in A$ and $b \in B$

  $$A \times B = \{ (a,b) \mid (a \in A) \land (b \in B) \}$$

- The Cartesian product is also known as the cross product

- **Definition**: A subset of a Cartesian product, $R \subseteq A \times B$ is called a relation. We will talk more about relations in the next set of slides

- **Note**: $A \times B \neq B \times A$ unless $A = \emptyset$ or $B = \emptyset$ or $A = B$. Find a counter example to prove this.
Cartesian Product (2)

- Cartesian Products can be generalized for any n-tuple

**Definition:** The Cartesian product of n sets, \( A_1, A_2, \ldots, A_n \), denoted \( A_1 \times A_2 \times \ldots \times A_n \), is

\[
A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \}
\]

\[
\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \ldots \times A_n
\]
Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language.
- Now we can simplify things using set notation!
- Example
  - $\forall x \in R \ (x^2 \geq 0)$
  - $\exists x \in Z \ (x^2 = 1)$
  - Also mixing quantifiers:
    \[ \forall a, b, c \in R \ \exists x \in C \ (ax^2 + bx + c = 0) \]
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Set Operations

• Arithmetic operators (+, -, ×, ÷) can be used on pairs of numbers to give us new numbers

• Similarly, set operators exist and act on two sets to give us new sets
  – Union \(\cup\)
  – Intersection \(\cap\)
  – Set difference \(\setminus\)
  – Set complement \(\overline{S}\)
  – Generalized union \(\bigcup\)
  – Generalized intersection \(\bigcap\)
Set Operators: Union

- **Definition**: The *union* of two sets $A$ and $B$ is the set that contains all elements in $A$, $B$, or both. We write:

$$A \cup B = \{ x \mid (x \in A) \lor (x \in B) \}$$
Set Operators: Intersection

• Definition: The intersection of two sets $A$ and $B$ is the set that contains all elements that are element of both $A$ and $B$. We write:

$$A \cap B = \{ x \mid (x \in A) \land (x \in B) \}$$
Disjoint Sets

• **Definition**: Two sets are said to be disjoint if their intersection is the empty set: \( A \cap B = \emptyset \)
Set Difference

• **Definition:** The *difference* of two sets $A$ and $B$, denoted $A \setminus B$ ($\setminus$) or $A - B$, is the set containing those elements that are in $A$ but not in $B$.
Set Complement

• **Definition:** The complement of a set $A$, denoted $\overline{A}$ ($\bar{A}$), consists of all elements not in $A$. That is the difference of the universal set and $A$: $U \setminus A$

$$\overline{A} = A^c = \{x \mid x \notin A\}$$
Set Complement: Absolute & Relative

• Given the Universe $U$, and $A, B \subset U$.
• The (absolute) complement of $A$ is $A=U\setminus A$
• The (relative) complement of $A$ in $B$ is $B\setminus A$
Set Identities

Let’s take a quick look at this Cheat Sheet or at Table 1 on page 130 in your textbook.

Table 3: Set Identities

<table>
<thead>
<tr>
<th>Expression</th>
<th>Identity laws</th>
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<tbody>
<tr>
<td>$A \cup \emptyset = A$</td>
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<td>$A \cap U = A$</td>
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<td>$A \cup A = A$</td>
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<td>$(A) = A$</td>
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<td>$A \cup B = B \cup A$</td>
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<td>$A \cap B = B \cap A$</td>
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<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
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<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
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<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td>Distributive laws</td>
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<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
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<td>$A \cup B = A \cap B$</td>
<td>De Morgan’s laws</td>
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<td>$A \cap B = \overline{A} \cup \overline{B}$</td>
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<td>$A \cup (A \cap B) = A$</td>
<td>Absorption laws</td>
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<td>$A \cap (A \cup B) = A$</td>
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<td>$A \cup \overline{A} = U$</td>
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<td>$A \cap \overline{A} = \emptyset$</td>
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Proving Set Equivalences

• Recall that to prove such identity, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal
• The book proves several of the standard set identities
• We will give a couple of different examples here
Proving Set Equivalences: Example A (1)

- Let
  - $A = \{x \mid x \text{ is even}\}$
  - $B = \{x \mid x \text{ is a multiple of 3}\}$
  - $C = \{x \mid x \text{ is a multiple of 6}\}$
- Show that $A \cap B = C$
Proving Set Equivalences: Example A (2)

\[ A \cap B \subseteq C: \quad \forall \ x \in A \cap B \]
\[ \Rightarrow x \text { is a multiple of } 2 \text { and } x \text { is a multiple of } 3 \]
\[ \Rightarrow \text { we can write } x = 2 \cdot 3 \cdot k \text { for some integer } k \]
\[ \Rightarrow x = 6k \text { for some integer } k \Rightarrow x \text { is a multiple of } 6 \]
\[ \Rightarrow x \in C \]

\[ C \subseteq A \cap B: \quad \forall \ x \in C \]
\[ \Rightarrow x \text { is a multiple of } 6 \Rightarrow x = 6k \text { for some integer } k \]
\[ \Rightarrow x = 2(3k) = 3(2k) \Rightarrow x \text { is a multiple of } 2 \text { and of } 3 \]
\[ \Rightarrow x \in A \cap B \]
Proving Set Equivalences: Example B (1)

• An alternative prove is to use membership tables where an entry is
  – 1 if a chosen (but fixed) element is in the set
  – 0 otherwise

• Example: Show that

\[ A \cap B \cap C = \overline{A} \cup \overline{B} \cup \overline{C} \]
Proving Set Equivalences: Example B (2)

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A∩B∩C</th>
<th>A∪B∪C</th>
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- 1 under a set indicates that “an element is in the set”
- If the columns are equivalent, we can conclude that indeed the two sets are equal
Generalizing Set Operations: Union and Intersection

• In the previous example, we showed De Morgan’s Law generalized to unions involving 3 sets
• In fact, De Morgan’s Laws hold for any finite set of sets
• Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets
Generalized Union

• **Definition:** The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n
\]

\[\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n\]
Generalized Intersection

• **Definition:** The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection.

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \]

LaTeX: \$\bigcap_{i=1}^{n}A_i=A_1\cap A_2 \ldots \cap A_n\$
Computer Representation of Sets (1)

• There really aren’t ways to represent infinite sets by a computer since a computer has a finite amount of memory.

• If we assume that the universal set $U$ is finite, then we can easily and effectively represent sets by bit vectors.

• Specifically, we force an ordering on the objects, say:

$$U = \{a_1, a_2, ..., a_n\}$$

• For a set $A \subseteq U$, a bit vector can be defined as, for $i=1,2,...,n$
  - $b_i=0$ if $a_i \notin A$
  - $b_i=1$ if $a_i \in A$
Computer Representation of Sets (2)

• Examples
  – Let U={0,1,2,3,4,5,6,7} and A={0,1,6,7}
  – The bit vector representing A is: 1100 0011
  – How is the empty set represented?
  – How is U represented?

• Set operations become trivial when sets are represented by bit vectors
  – Union is obtained by making the bit-wise OR
  – Intersection is obtained by making the bit-wise AND
Computer Representation of Sets (3)

• Let U={0,1,2,3,4,5,6,7}, A={0,1,6,7}, B={0,4,5}
• What is the bit-vector representation of B?
• Compute, bit-wise, the bit-vector representation of A∩B
• Compute, bit-wise, the bit-vector representation of A∪B
Programming Question

• Using bit vector, we can represent sets of cardinality equal to the size of the vector
• What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?
• What data structure could we use?