Functions

Section 2.3 of Rosen
Spring 2018
CSCE 235H Introduction to Discrete Structures (Honors)
Course web-page: cse.unl.edu/~cse235h
Questions: Piazza
Outline

• Definitions & terminology
  – function, domain, co-domain, image, preimage (antecedent), range, image of a set, strictly increasing, strictly decreasing, monotonic

• Properties
  – One-to-one (injective)
  – Onto (surjective)
  – One-to-one correspondence (bijective)
  – Exercises (5)

• Inverse functions (examples)

• Operators
  – Composition, Equality

• Important functions
  – identity, absolute value, floor, ceiling, factorial
Introduction

• You have already encountered function
  – \( f(x,y) = x+y \)
  – \( f(x) = x \)
  – \( f(x) = \sin(x) \)

• Here we will study functions defined on discrete domains and ranges

• We may not always be able to write function in a ‘neat way’ as above
Definition: Function

- **Definition**: A function \( f \) from a set \( A \) to a set \( B \) is an assignment of **exactly one** element of \( B \) to **each** element of \( A \).

- We write \( f(a) = b \) if \( b \) is the unique element of \( B \) assigned by the function \( f \) to the element \( a \in A \).

- **Notation**: \( f: A \rightarrow B \) which can be read as ‘\( f \) maps \( A \) to \( B \)’

- **Note the subtlety**
  - Each and every element of \( A \) has a **single** mapping
  - Each element of \( B \) may be mapped to by **several** elements in \( A \) or **not at all**
Terminology

- Let $f: A \rightarrow B$ and $f(a)=b$. Then we use the following terminology:
  - $A$ is the **domain** of $f$, denoted $\text{dom}(f)$
  - $B$ is the **co-domain** of $f$
  - $b$ is the **image** of $a$
  - $a$ is the **preimage** (antecedent) of $b$
  - The **range** of $f$ is the set of all images of elements of $A$, denoted $\text{rng}(f)$
Function: Visualization

A function, $f: A \rightarrow B$

- **Domain**: A
- **Co-Domain**: B
- **Range**: $f(a) = b$
- **Image**: $f(a) = b$
- **Preimage**: A
More Definitions (1)

• **Definition:** Let $f_1$ and $f_2$ be two functions from a set $A$ to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from $A$ to $\mathbb{R}$ defined by:
  
  $$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
  $$(f_1 f_2)(x) = f_1(x)f_2(x)$$

• **Example:** Let $f_1(x) = x^4 + 2x^2 + 1$ and $f_2(x) = 2 - x^2$
  
  $$(f_1 + f_2)(x) = x^4 + 2x^2 + 1 + 2 - x^2 = x^4 + x^2 + 3$$
  $$(f_1 f_2)(x) = (x^4 + 2x^2 + 1)(2 - x^2) = -x^6 + 3x^2 + 2$$
More Definitions (2)

• **Definition**: Let $f: A \rightarrow B$ and $S \subseteq A$. The **image of the set $S$** is the subset of $B$ that consists of all the images of the elements of $S$. We denote the image of $S$ by $f(S)$, so that

$$f(S)=\{ f(s) \mid \forall \ s \in S \}$$

• Note there that the image of $S$ is a set and not an element.
Image of a set: Example

• Let:
  – A = \{a_1, a_2, a_3, a_4, a_5\}
  – B = \{b_1, b_2, b_3, b_4, b_5\}
  – f = \{(a_1, b_2), (a_2, b_3), (a_3, b_3), (a_4, b_1), (a_5, b_4)\}
  – S = \{a_1, a_3\}

• Draw a diagram for f

• What is the:
  – Domain, co-domain, range of f?
  – Image of S, f(S)?
More Definitions (3)

- **Definition**: A function $f$ whose domain and codomain are subsets of the set of real numbers ($\mathbb{R}$) is called
  - **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and $x$ and $y$ are in the domain of $f$.
  - **strictly decreasing** if $f(x) > f(y)$ whenever $x < y$ and $x$ and $y$ are in the domain of $f$.

- A function that is increasing or decreasing is said to be **monotonic**
Outline

• Definitions & terminology

• **Properties**
  – One-to-one (injective)
  – Onto (surjective)
  – One-to-one correspondence (bijective)
  – Exercises (5)

• Inverse functions (examples)

• Operators

• Important functions
Definition: Injection

• **Definition**: A function \( f \) is said to be **one-to-one** or **injective** (or an injection) if

\[
\forall \ x \text{ and } y \text{ in in the domain of } f, \ f(x) = f(y) \implies x = y
\]

• Intuitively, an injection simply means that each element in the range has **at most** one preimage (antecedent)
• It is useful to think of the contrapositive of this definition

\[
x \neq y \implies f(x) \neq f(y)
\]
Definition: Surjection

- **Definition**: A function $f: A \rightarrow B$ is called **onto** or **surjective** (or an surjection) if
  \[
  \forall \ b \in B, \ \exists \ a \in A \text{ with } f(a)=b
  \]
- Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)
- Thus, the range is the same as the codomain
Definition: Bijection

- **Definition**: A function $f$ is a one-to-one correspondence (or a bijection), if it is both
  - one-to-one (injective) and
  - onto (surjective)

- One-to-one correspondences are important because they endow a function with an inverse.
- They also allow us to have a concept cardinality for infinite sets
- Let’s look at a few examples to develop a feel for these definitions...
**Functions: Example 1**

- Is this a function? Why?
- No, because each of $a_1$, $a_2$ has two images
Is this a function

- One-to-one (injective)? Why? No, $b_1$ has 2 preimages
- Onto (surjective)? Why? No, $b_4$ has no preimage
Functions: Example 3

- Is this a function
  - One-to-one (injective)? Why? Yes, no \( b_i \) has 2 preimages
  - Onto (surjective)? Why? No, \( b_4 \) has no preimage
Functions: Example 4

- Is this a function
  - One-to-one (injective)? Why? No, $b_3$ has 2 preimages
  - Onto (surjective)? Why? Yes, every $b_i$ has a preimage
Functions: Example 5

- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence.
Exercice 1

• Let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) be defined by
  \[
  f(x) = 2x - 3
  \]

• What is the domain, codomain, range of \( f \)?

• Is \( f \) one-to-one (injective)?

• Is \( f \) onto (surjective)?

• Clearly, \( \text{dom}(f) = \mathbb{Z} \). To see what the range is, note that:
  \[
  b \in \text{rng}(f) \iff b = 2a - 3, \quad \text{with } a \in \mathbb{Z}
  \]
  \[
  \iff b = 2(a - 2) + 1
  \]
  \[
  \iff b \text{ is odd}
  \]
Exercise 1 (cont’ d)

• Thus, the range is the set of all odd integers
• Since the range and the codomain are different (i.e., \( \text{rng}(f) \neq \mathbb{Z} \)), we can conclude that \( f \) is not onto (surjective)
• However, \( f \) is one-to-one injective. Using simple algebra, we have:
  \[
f(x_1) = f(x_2) \implies 2x_1 - 3 = 2x_2 - 3 \implies x_1 = x_2 \quad \text{QED}
\]
Exercise 2

• Let $f$ be as before

$$f(x) = 2x - 3$$

but now we define $f: \mathbb{N} \rightarrow \mathbb{N}$

• What is the domain and range of $f$?

• Is $f$ onto (surjective)?

• Is $f$ one-to-one (injective)?

• By changing the domain and codomain of $f$, $f$ is not even a function anymore. Indeed, $f(1) = 2 \cdot 1 - 3 = -1 \notin \mathbb{N}$
Exercice 3

• Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(x) = x^2 - 5x + 5$

• Is this function
  – One-to-one?
  – Onto?
Exercice 3: Answer

• It is not one-to-one (injective)

\[ f(x_1) = f(x_2) \Rightarrow x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2 \]
\[ \Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2) \]
\[ \Rightarrow (x_1 + x_2) = 5 \]

Many \( x_1, x_2 \in \mathbb{Z} \) satisfy this equality. There are thus an infinite number of solutions. In particular, \( f(2) = f(3) = -1 \)

• It is also not onto (surjective).

The function is a parabola with a global minimum at \((5/2, -5/4)\). Therefore, the function fails to map to any integer less than -1

• What would happen if we changed the domain/codomain?
Exercice 4

• Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by
  
  \[ f(x) = 2x^2 + 7x \]

• Is this function
  – One-to-one (injective)?
  – Onto (surjective)?

• Again, this is a parabola, it cannot be onto
  (where is the global minimum?)
Exercice 4: Answer

• **f(x) is one-to-one! Indeed:**
  
  \[ f(x_1) = f(x_2) \Rightarrow 2x_1^2 + 7x_1 = 2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1 \]
  
  \[ \Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 + x_2) = -7/2 \]

  But \(-7/2 \notin \mathbb{Z}\). Therefore it must be the case that \(x_1 = x_2\).

  It follows that \(f\) is a one-to-one function. \hspace{1cm} \text{QED} \\

• **f(x) is not surjective because \(f(x)=1\) does not exist**

  \[ 2x^2 + 7x = 1 \Rightarrow x(2x + 7) = 1 \]

  The product of two integers is 1 if both integers are 1 or -1.

  \[ x = 1 \Rightarrow (2x + 7) = 1 \Rightarrow 9 = 1, \text{ impossible} \]

  \[ x = -1 \Rightarrow -(2+7) = 1 \Rightarrow -5 = 1, \text{ impossible} \]
Exercise 5

• Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by
  \[ f(x) = 3x^3 - x \]

• Is this function
  – One-to-one (injective)?
  – Onto (surjective)?
Exercice 5: $f$ is one-to-one

- To check if $f$ is one-to-one, again we suppose that for $x_1, x_2 \in \mathbb{Z}$ we have $f(x_1) = f(x_2)$

  $f(x_1) = f(x_2) \Rightarrow 3x_1^3 - x_1 = 3x_2^3 - x_2$

  $\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$

  $\Rightarrow 3(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$

  $\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = 1/3$

  which is impossible because $x_1, x_2 \in \mathbb{Z}$

thus, $f$ is one-to-one
**Exercice 5: \( f \) is not onto**

- Consider the counter example \( f(a) = 1 \)
- If this were true, we would have
  \[
  3a^3 - a = 1 \implies a(3a^2 - 1) = 1 \text{ where } a \text{ and } (3a^2 - 1) \in \mathbb{Z}
  \]
- The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1
- Neither 1 nor -1 satisfy the above equality
  - Thus, we have identified \( 1 \in \mathbb{Z} \) that does not have an antecedent and \( f \) is not onto (surjective)
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• Properties
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  – Composition, Equality

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Inverse Functions (1)

• **Definition:** Let $f: A \rightarrow B$ be a bijection. The inverse function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$

• The inverse function is denote $f^{-1}$.

• When $f$ is a bijection, its inverse exists and

$$f(a) = b \iff f^{-1}(b) = a$$
Inverse Functions (2)

• Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is invertible.

• Why must a function be bijective to have an inverse?
  – Consider the case where f is not one-to-one (not injective). This means that some element b ∈ B has more than one antecedent in A, say a₁ and a₂. How can we define an inverse? Does f⁻¹(b) = a₁ or a₂?
  – Consider the case where f is not onto (not surjective). This means that there is some element b ∈ B that does not have any preimage a ∈ A. What is then f⁻¹(b)?
Inverse Functions: Representation

A function and its inverse

Domain

Co-Domain
Inverse Functions: Example 1

- Let $f: \mathbb{R} \to \mathbb{R}$ be defined by
  \[ f(x) = 2x - 3 \]
- What is $f^{-1}$?
  
  1. We must verify that $f$ is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.
  
  2. To find the inverse, we use the substitution
     
     - Let $f^{-1}(y) = x$
     - And $y = 2x - 3$, which we solve for $x$. Clearly, $x = (y+3)/2$
     - So, $f^{-1}(y) = (y+3)/2$
Inverse Functions: Example 2

- Let $f(x) = x^2$. What is $f^{-1}$?
- No domain/codomain has been specified.
- Say $f: \mathbb{R} \to \mathbb{R}$
  - Is $f$ a bijection? Does its inverse exist?
  - Answer: No
- Say we specify that $f: A \to B$ where
  $$ A = \{x \in \mathbb{R} \mid x \leq 0\} \text{ and } B = \{y \in \mathbb{R} \mid y \geq 0\} $$
  - Is $f$ a bijection? Does its inverse exist?
  - Answer: Yes, the function becomes a bijection and thus, has an inverse
Inverse Functions: Example 2 (cont’)

• To find the inverse, we let
  – \( f^{-1}(y) = x \)
  – \( y = x^2 \), which we solve for \( x \)

• Solving for \( x \), we get \( x = \pm \sqrt{y} \), but which one is it?

• Since \( \text{dom}(f) \) is all nonpositive and \( \text{rng}(f) \) is nonnegative, thus \( x \) must be nonpositive and

\[
f^{-1}(y) = -\sqrt{y}
\]

• From this, we see that the domains/codomains are just as important to a function as the definition of the function itself
Inverse Functions: Example 3

• Let \( f(x) = 2^x \)
  – What should the domain/codomain be for this function to be a bijection?
  – What is the inverse?

• The function should be \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \)

• Let \( f^{-1}(y) = x \) and \( y = 2^x \), solving for \( x \) we get \( x = \log_2(y) \).
  Thus, \( f^{-1}(y) = \log_2(y) \)

• What happens when we include 0 in the codomain?
• What happens when restrict either sets to \( \mathbb{Z} \)?
Function Composition (1)

• The value of functions can be used as the input to other functions

• **Definition:** Let $g: A \rightarrow B$ and $f: B \rightarrow C$. The composition of the functions $f$ and $g$ is

\[
(f \circ g)(x) = f(g(x))
\]

• $f \circ g$ is read as ‘$f$ circle $g’’, or ‘$f$ composed with $g’’, ‘$f$ following $g’’, or just ‘$f$ of $g’’

• In LaTeX: $\circ$
Function Composition (2)

• Because \((f \circ g)(x) = f(g(x))\), the composition \(f \circ g\) cannot be defined unless the range of \(g\) is a subset of the domain of \(f\)

\[ f \circ g \text{ is defined } \iff \text{rng}(g) \subseteq \text{dom}(f) \]

• The order in which you apply a function matters: you go from the inner most to the outer most

• It follows that \(f \circ g\) is in general not the same as \(g \circ f\)
Composition: Graphical Representation

The composition of two functions

\[(f \circ g)(a)\]

- Domain of \(g\) and \(a\)
- Co-domain of \(g\) and \(g(a)\)
- Range of \(g\) and \(rng(g)\)
- Domain of \(f\) and \(f(g(a))\)
Composition: Graphical Representation

The composition of two functions

\[(f \circ g)(a)\]
Composition: Example 1

• Let \( f, g \) be two functions on \( R \rightarrow R \) defined by
  \[
  f(x) = 2x - 3 \\
  g(x) = x^2 + 1
  \]

• What are \( f \circ g \) and \( g \circ f \)?

• We note that
  
  – \( f \) is bijective, thus \( \text{dom}(f) = \text{rng}(f) = \text{codomain}(f) = R \)
  
  – For \( g \), \( \text{dom}(g) = R \) but \( \text{rng}(g) = \{x \in R \mid x \geq 1\} \subseteq R^+ \)
  
  – Since \( \text{rng}(g) = \{x \in R \mid x \geq 1\} \subseteq R^+ \subseteq \text{dom}(f) = R \), \( f \circ g \) is defined
  
  – Since \( \text{rng}(f) = R \subseteq \text{dom}(g) = R \), \( g \circ f \) is defined
Composition: Example 1 (cont’)

- Given \( f(x) = 2x - 3 \) and \( g(x) = x^2 + 1 \)
- \((f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3 = 2x^2 - 1\)
- \((g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 +1 = 4x^2 - 12x + 10\)
Function Equality

• Although it is intuitive, we formally define what it means for two functions to be equal

• **Lemma**: Two functions $f$ and $g$ are equal if and only
  
  – $\text{dom}(f) = \text{dom}(g)$
  
  – $\forall a \in \text{dom}(f) \ (f(a) = g(a))$
Associativity

• The composition of function is not commutative \((f \circ g \neq g \circ f)\), it is associative

• **Lemma**: The composition of functions is an associative operation, that is

\[
(f \circ g) \circ h = f \circ (g \circ h)
\]
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Important Functions: Identity

- **Definition**: The *identity* function on a set $A$ is the function

$$\iota: A \rightarrow A$$

defined by $\iota(a) = a$ for all $a \in A$.

- One can view the identity function as a composition of a function and its inverse:

$$\iota(a) = (f \circ f^{-1})(a) = (f^{-1} \circ f)(a)$$

- Moreover, the composition of any function $f$ with the identity function is itself $f$:

$$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$
Inverses and Identity

• The identity function, along with the composition operation, gives us another characterization of inverses when a function has an inverse.

• **Theorem**: The functions $f: A \rightarrow B$ and $g: B \rightarrow A$ are inverses if and only if

$$ (g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B $$

where the $\iota_A$ and $\iota_B$ are the identity functions on sets $A$ and $B$. That is,

$$ \forall a \in A, b \in B \left( (g(f(a)) = a) \land (f(g(b)) = b) \right) $$
Important Functions: Absolute Value

• **Definition:** The absolute value function, denoted $|x|$, $f: \mathbb{R} \rightarrow \{y \in \mathbb{R} \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x \leq 0 
\end{cases}$$
Important Functions: Floor & Ceiling

• Definitions:
  – The **floor function**, denoted \( \lfloor x \rfloor \), is a function \( R \rightarrow \mathbb{Z} \). Its values is the **largest integer** that is less than or equal to \( x \)
  – The ceiling function, denoted \( \lceil x \rceil \), is a function \( R \rightarrow \mathbb{Z} \). Its values is the **smallest integer** that is greater than or equal to \( x \)
• In LaTeX: \( \lceil \), \( \rceil \), \( \lfloor \), \( \rfloor \)
Important Functions: Floor
Important Functions: Ceiling
Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects.

- **Definition:** The **factorial** function, denoted $n!$, is a function $\mathbb{N} \rightarrow \mathbb{N}^+$. Its value is the **product** of the n positive integers.

\[
 n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n - 1) \cdot n
\]
Factorial Function & Stirling’s Approximation

• The factorial function is defined on a discrete domain
• In many applications, it is useful a continuous version of the function (say if we want to differentiate it)
• To this end, we have the Stirling’s formula

\[ n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]
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