Predicate Logic and Quantifies

Sections 1.4, and 1.5 of Rosen
Spring 2017
CSCE 235H Introduction to Discrete Structures (Honors)
Course web-page: cse.unl.edu/~cse235h
All questions: Piazza
LaTeX

• Using the package: \usepackage{amssymb}
  – Set of natural numbers: $\mathbb{N}$
  – Set of integer numbers: $\mathbb{Z}$
  – Set of rational numbers: $\mathbb{Q}$
  – Set of real numbers: $\mathbb{R}$
  – Set of complex numbers: $\mathbb{C}$
Outline

• Introduction
• Terminology:
  – Propositional functions; arguments; arity; universe of discourse
• Quantifiers
  – Definition; using, mixing, negating them
• Logic Programming (Prolog)
• Transcribing English to Logic
• More exercises
Introduction

• Consider the statements:
  \[ x > 3, \ x = y + 3, \ x + y = z \]

• The symbols >, +, = denote relations between \( x \) and 3, \( x \), \( y \), and 4, and \( x, y \), and \( z \), respectively.

• These relations may hold or not hold depending on the values that \( x, y, \) and \( z \) may take.

• A **predicate** is a property that is affirmed or denied about the subject (in logic, we say ‘**variable**’ or ‘**argument**’) of a statement.

• Consider the statement: ‘\( x \) is greater than 3’
  – ‘\( x \)’ is the subject
  – ‘is greater than 3’ is the predicate.
Propositional Functions (1)

• To write in Predicate Logic ‘x is greater than 3’
  – We introduce a functional symbol for the **predicate** and
  – Put the subject as an **argument** (to the functional symbol): $P(x)$

• Terminology
  – $P(x)$ is a statement
  – $P$ is a predicate or propositional function
  – $x$ as an argument
  – $P($Bob$)$ is a proposition
Propositional Functions (2)

• Examples:
  – Father(x): unary predicate
  – Brother(x,y): binary predicate
  – Sum(x,y,z): ternary predicate
  – P(x,y,z,t): n-ary predicate
Propositional Functions (3)

• **Definition:** A statement of the form \( P(x_1, x_2, \ldots, x_n) \) is the value of the propositional symbol \( P \).

• **Here:** \((x_1, x_2, \ldots, x_n)\) is an \( n \)-tuple and \( P \) is a predicate

• **We can think of a propositional function as a function that**
  – Evaluates to true or false
  – Takes one or more arguments
  – Expresses a predicate involving the argument(s)
  – Becomes a *proposition* when values are assigned to the arguments
Propositional Functions: Example

• Let $Q(x,y,z)$ denote the statement ‘$x^2+y^2=z^2$’
  – What is the truth value of $Q(3,4,5)$?
    $Q(3,4,5)$ is true
  – What is the truth value of $Q(2,2,3)$?
    $Q(2,3,3)$ is false
  – How many values of $(x,y,z)$ make the predicate true?
    There are infinitely many values that make the proposition true, how many right triangles are there?
Universe of Discourse

• Consider the statement ‘x>3’, does it make sense to assign to x the value ‘blue’?
• Intuitively, the universe of discourse is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
• What would be the universe of discourse for the propositional function below be:
  \[ \text{EnrolledCSE235}(x) = \text{‘x is enrolled in CSE235’} \]
Universe of Discourse: Multivariate functions

• Each variable in an $n$-tuple (i.e., each argument) may have a different universe of discourse

• Consider an $n$-ary predicate $P$:
  \[ P(r,g,b,c) = \text{‘The rgb-values of the color c is } (r,g,b)’ \]

• Example, what is the truth value of
  – $P(255,0,0,\text{red})$
  – $P(0,0,255,\text{green})$

• What are the universes of discourse of $(r,g,b,c)$?
Alert

• Propositional Logic (PL)
  – Sentential logic
  – Boolean logic
  – Zero order logic

• First Order Logic (FOL)
  – Predicate logic (PL)
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Quantifiers: Introduction

• The statement ‘\(x>3\)’ is not a proposition

• It becomes a proposition
  – When we assign values to the argument: ‘\(4>3\)’ is true, ‘\(2<3\)’ is false,
  or
  – When we quantify the statement

• Two quantifiers
  – Universal quantifier \(\forall\)
    the proposition is true for all possible values in the universe of discourse
  – Existential quantifier \(\exists\)
    the proposition is true for some value(s) in the universe of discourse
Universal Quantifier: Definition

- **Definition:** The universal quantification of a predicate $P(x)$ is the proposition ‘$P(x)$ is true for all values of $x$ in the universe of discourse.’ We use the notation: $\forall x P(x)$, which is read ‘for all $x$’.
- If the universe of discourse is finite, say $\{n_1, n_2, ..., n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements
  \[ \forall x P(x) \iff P(n_1) \land P(n_2) \land ... \land P(n_k) \]
Universal Quantifier: Example 1

• Let
  – $P(x)$: ‘$x$ must take a discrete mathematics course’ and
  – $Q(x)$: ‘$x$ is a CS student.’

• The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.

• Express the statements:
  – “Every CS student must take a discrete mathematics course.”
    \[ \forall x \ Q(x) \rightarrow P(x) \]
  – “Everybody must take a discrete mathematics course or be a CS student.”
    \[ \forall x \ ( P(x) \lor Q(x) ) \]
  – “Everybody must take a discrete mathematics course and be a CS student.”
    \[ \forall x \ ( P(x) \land Q(x) ) \]

Are these statements true or false at UNL?
Universal Quantifier: Example 2

• Express in FOL the statement
  ‘for every $x$ and every $y$, $x+y>10$’

• Answer:
  1. Let $P(x,y)$ be the statement $x+y>10$
  2. Where the universe of discourse for $x$, $y$ is the set of integers
  3. The statement is: $\forall x \forall y \ P(x,y)$

• Shorthand: $\forall x, y \ P(x,y)$
Existential Quantifier: Definition

- **Definition**: The existential quantification of a predicate $P(x)$ is the proposition ‘There exists a value $x$ in the universe of discourse such that $P(x)$ is true’
  - Notation: $\exists x \ P(x)$
  - Reads: ‘there exists $x$’

- If the universe of discourse is finite, say $\{n_1,n_2,\ldots,n_k\}$, then the existential quantifier is simply the disjunction of the propositions over all the elements

  $$\exists x \ P(x) \iff P(n_1) \lor P(n_2) \lor \ldots \lor P(n_k)$$
Existential Quantifier: Example 1

• Let $P(x,y)$ denote the statement ‘$x+y=5$’
• What does the expression $\exists x \exists y P(x,y)$ mean?
• Which universe(s) of discourse make it true?
Existential Quantifier: Example 2

- Express formally the statement
  ‘there exists a real solution to \( ax^2 + bx - c = 0 \)’

- Answer:
  1. Let \( P(x) \) be the statement \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
  2. Where the universe of discourse for \( x \) is the set of real numbers. Note here that \( a, b, c \) are fixed constants.
  3. The statement can be expressed as \( \exists x \ P(x) \)

- What is the truth value of \( \exists x \ P(x) \), where UoD is \( \mathbb{R} \)?
  - It is false. When \( b^2 < 4ac \), there are no real number \( x \) that can satisfy the predicate

- What can we do so that \( \exists x \ P(x) \) is true?
  - Change the universe of discourse to the complex numbers, \( \mathbb{C} \)
Quantifiers: Truth values

- In general, when are quantified statements true or false?

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<td>$\forall x \ P(x)$</td>
<td>$P(x)$ is true for every $x$</td>
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Mixing quantifiers (1)

- Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

\[ \forall x \exists y \ P(x,y) \]

is perfectly valid

- Alert:
  - The quantifiers must be read from left to right
  - The order of the quantifiers is important
  - \[ \forall x \exists y \ P(x,y) \] is not equivalent to \[ \exists y \forall x \ P(x,y) \]
Mixing quantifiers (2)

• Consider
  – $\forall x \exists y \text{ Loves}(x,y)$: Everybody loves somebody
  – $\exists y \forall x \text{ Loves}(x,y)$: There is someone loved by everyone

• The two expressions do not mean the same thing

• $(\exists y \forall x \text{ Loves}(x,y)) \rightarrow (\forall x \exists y \text{ Loves}(x,y))$
  but the converse does not hold

• However, you can commute similar quantifiers
  – $\forall x \forall y P(x,y)$ is equivalent to $\forall y \forall x P(x,y)$ (thus, $\forall x, y P(x,y)$)
  – $\exists x \exists y P(x,y)$ is equivalent to $\exists y \exists x P(x,y)$ (thus $\exists x, y P(x,y)$)
# Mixing Quantifiers: Truth values

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<td>$P(x,y)$ is true for every pair $x,y$</td>
<td>There is at least one pair $x,y$ for which $P(x,y)$ is false</td>
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<tr>
<td>$\forall x \exists y , P(x,y)$</td>
<td>For every $x$, there is a $y$ for which $P(x,y)$ is true</td>
<td>There is an $x$ for which $P(x,y)$ is false for every $y$</td>
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Mixing Quantifiers: Example (1)

• Express, in predicate logic, the statement that there is an infinite number of integers

• Answer:

1. Let $P(x,y)$ be the statement that $x < y$
2. Let the universe of discourse be the integers, $\mathbb{Z}$
3. The statement can be expressed by the following

   $\forall x \exists y P(x,y)$
Mixing Quantifiers: Example (2)

• Express the *commutative law of addition* for $R$

• We want to express that for every pair of reals, $x, y$, the following holds: $x+y = y+x$

• Answer:
  1. Let $P(x, y)$ be the statement that $x+y$
  2. Let the universe of discourse be the reals, $R$
  3. The statement can be expressed by the following

\[ \forall x \forall y (P(x, y) \iff P(y, x)) \]

Alternatively, \( \forall x \forall y (x+y = y+x) \)
Mixing Quantifiers: Example (3)

• Express the multiplicative law for nonzero reals $R \setminus \{0\}$ (i.e., every nonzero real has an inverse)
• We want to express that for every real number $x$, there exists a real number $y$ such that $xy=1$
• Answer:

$$\forall x \exists y (xy = 1)$$
Mixing Quantifiers: Example (4)
false mathematical statement

- Does commutativity for subtraction hold over the reals?
- That is: does $x-y = y-x$ for all pairs $x, y$ in $\mathbb{R}$?
- Express using quantifiers

$$\forall x \ \forall y \ (x-y = y-x)$$
Mixing Quantifiers: Example (5)

• Express the statement as a logical expression:
  – “There is a number \( x \) such that
  – when it is added to any number, the result is that number and
  – if it is multiplied by any number, the result is \( x \)”

• Answer:
  • Let \( P(x,y) \) be the expression “\( x+y=y \)”
  • Let \( Q(x,y) \) be the expression “\( xy=x \)”
  • The universe of discourse is \( N,Z,R,Q \) (but not \( Z^+ \))
  • Then the expression is:
    \[
    \exists x \ \forall y \ P(x,y) \land Q(x,y)
    \]
    Alternatively: \[
    \exists x \ \forall y \ (x+y=y) \land (xy = x)
    \]
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Binding Variables

• When a quantifier is used on a variable $x$, we say that $x$ is bound
• If no quantifier is used on a variable in a predicate statement, the variable is called free
• Examples
  – In $\exists x \forall y P(x,y)$, both $x$ and $y$ are bound
  – In $\forall x P(x,y)$, $x$ is bound but $y$ is free
• A statement is called a well-formed formula, when all variables are properly quantified
Binding Variables: Scope

• The set of all variables bound by a common quantifier is called the scope of the quantifier.
• For example, in the expression $\exists x, y \forall z P(x, y, z, c)$
  – What is the scope of existential quantifier?
  – What is the scope of universal quantifier?
  – What are the bound variables?
  – What are the free variables?
  – Is the expression a well-formed formula?
Negation

- We can use negation with quantified expressions as we used them with propositions
- **Lemma**: Let $P(x)$ be a predicate. Then the followings hold:

\[
\neg (\forall x \ P(x)) \equiv \exists x \ \neg P(x)
\]

\[
\neg (\exists x \ P(x)) \equiv \forall x \ \neg P(x)
\]

- This is essentially the quantified version of De Morgan’s Law (when the universe of discourse is finite, this is exactly De Morgan’s Law)
## Negation: Truth

### Truth Values of Negated Quantifiers

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<td>$\neg \exists x \ P(x) \equiv \forall x \ \neg P(x)$</td>
<td>$P(x)$ is false for every $x$</td>
<td>There is an $x$ for which $P(x)$ is true</td>
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<td>$\neg \forall x \ P(x) \equiv \exists x \ \neg P(x)$</td>
<td>There is an $x$ for which $P(x)$ is false</td>
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Negation: Example

- Rewrite the following expression, pushing negation inward:
  \[ \neg \forall x (\exists y \forall z P(x,y,z) \land \exists z \forall y P(x,y,z)) \]

- Answer:
  \[ \exists x (\forall y \exists z \neg P(x,y,z) \lor \forall z \exists y \neg P(x,y,z)) \]
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Prolog (1)

• Prolog (Programming in Logic)
  – is a programming language
  – based on (a restricted form of) Predicate Logic
    (a.k.a. Predicate Calculus and FOL)

• It was developed
  – by the logicians of the Artificial Intelligence community
  – for symbolic reasoning
• Prolog allows the users to express facts and rules
• Facts are propositional functions:
  – student(mia),
  – enrolled(mia,cse235),
  – instructor(patel,cse235), etc.
• Rules are implications with conjunctions:
  teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
• Prolog answers queries such as:
  ?enrolled(mia,cse235)
  ?enrolled(X,cse476)
  ?teaches(X,mia)
  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5
English into Logic

• Logic is more precise than English
• Transcribing English into Logic and vice versa can be tricky
• When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:
  
  **Use ∀ with ⇒**
  \[ \forall x \text{ Lion}(x) \Rightarrow \text{Fierce}(x) \]  
  Every lion is fierce
  \[ \forall x \text{ Lion}(x) \land \text{Fierce}(x) \]  
  Everyone is a lion and everyone is fierce

  **Use ∃ with ∧**
  \[ \exists x \text{ Lion}(x) \land \text{Vegan}(x) \]  
  Holds when you have at least one vegan lion
  \[ \exists x \text{ Lion}(x) \Rightarrow \text{Vegan}(x) \]  
  Holds when you have vegan people in the universe of discourse (even though there is no vegan lion in the universe of discourse)
More Exercises (1)

• Let $P(x,y)$ denote ‘$x$ is a factor of $y$’ where
  – $x \in \{1,2,3,...\}$ and $y \in \{2,3,4,...\}$

• Let $Q(x,y)$ denote:
  – $\forall x, y \ [P(x,y) \rightarrow (x=y) \lor (x=1)]$

• Question: When is $Q(x,y)$ true?
Alert...

• Some students wonder if:
  \[ \forall x,y \ P(x,y) \equiv (\forall x \ P(x,y)) \land (\forall y \ P(x,y)) \]

• This is certainly not true.
  – In the left-hand side, both \(x,y\) are bound.
  – In the right-hand side,
    • In the first predicate, \(x\) is bound and \(y\) is free
    • In the second predicate, \(y\) is bound and \(x\) is free
    • Thus, the left-hand side is a proposition, but the right-hand side is not. They cannot be equivalent

• All variables that occur in a propositional function must be bound to turn it into a proposition