Combinatorics

Section 6.1—6.6  8.5—8.6 of Rosen
Spring 2017
CSCE 235H Introduction to Discrete Structures (Honors)
Course web-page: cse.unl.edu/~cse235h
Questions: Piazza
Motivation

• Combinatorics is the study of collections of objects. Specifically, counting objects, arrangement, derangement, etc. along with their mathematical properties

• Counting objects is important in order to analyze algorithms and compute discrete probabilities

• Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability
  – A simple example: How many arrangements are there of a deck of 52 cards?

• In addition, combinatorics can be used as a proof technique
  – A combinatorial proof is a proof method that uses counting arguments to prove a statement
Outline

• Introduction
• Counting:
  – Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  – Application of PIE: Number of onto functions
• Pigeonhole principle
  – Generalized, probabilistic forms
• Permutations
• Combinations
• Binomial Coefficients
• Generalizations
  – Combinations with repetitions, permutations with indistinguishable objects
• Algorithms
  – Generating combinations (1), permutations (2)
• More Examples
Product Rule

- If two events are not mutually exclusive (that is we do them separately), then we apply the product rule.
- **Theorem: Product Rule**
  
  Suppose a procedure can be accomplished with two disjoint subtasks. If there are
  
  - $n_1$ ways of doing the first task and
  - $n_2$ ways of doing the second task,
  
  then there are $n_1 \cdot n_2$ ways of doing the overall procedure.
Sum Rule (1)

- If two events are mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule.

- **Theorem:** Sum Rule. If
  
  - an event $e_1$ can be done in $n_1$ ways,
  
  - an event $e_2$ can be done in $n_2$ ways, and
  
  - $e_1$ and $e_2$ are mutually exclusive

  then the number of ways of both events occurring is $n_1 + n_2$.
Sum Rule (2)

• There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur
  \[ n_1 + n_2 + \ldots + n_{m-1} + n_m \]

• We can give another formulation in terms of sets. Let \( A_1, A_2, \ldots, A_m \) be pairwise disjoint sets. Then
  \[ |A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| \cup |A_2| \cup \ldots \cup |A_m| \]

(In fact, this is a special case of the general Principal of Inclusion-Exclusion (PIE))
Principle of Inclusion-Exclusion (PIE)

• Say there are two events, $e_1$ and $e_2$,
  – For which there are $n_1$ and $n_2$ possible outcomes respectively.
  – But, some outcome $n_i$ could result from $e_1$ and also from $e_2$
• Now, say that only one event can occur, not both
• In this situation, we cannot apply the sum rule. Why?
  ... because we would be over counting the number of possible outcomes.
• Instead we have to count the number of possible outcomes of $e_1$ and $e_2$
  minus the number of possible outcomes in common to both; i.e., the
  number of ways to do both tasks
• If again we think of them as sets, we have
  $$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
More generally, we have the following

**Lemma:** Let $A$, $B$, be subsets of a finite set $U$. Then

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cap B| \leq \min \{|A|, |B|\}$
3. $|A \backslash B| = |A| - |A \cap B| \geq |A| - |B|$
4. $|\overline{A}| = |U| - |A|$
5. $|A \oplus B| = |A \cup B| - |A \cap B|$
   $= |A| + |B| - 2|A \cap B| = |A \backslash B| + |B \backslash A|$
6. $|A \times B| = |A| \times |B|$
PIE: Theorem

**Theorem:** Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Each summation is over

- all $i$,
- pairs $i,j$ with $i<j$,
- triples with $i<j<k$, etc.
PIE Theorem: Example 1

- To illustrate, when \( n=3 \), we have

\[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|
\]
PIE Theorem: Example 2

• To illustrate, when n=4, we have

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| \]

\[ - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \]

\[ + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \]

\[ - |A_1 \cap A_2 \cap A_3 \cap A_4| \]
Application of PIE: Example A (1)

- How many integers between 1 and 300 (inclusive) are
  - Divisible by at least one of 3, 5, 7?
  - Divisible by 3 and by 5 but not by 7?
  - Divisible by 5 but by neither 3 or 7?

- Let
  \[ A = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (3 \mid n) \} \]
  \[ B = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (5 \mid n) \} \]
  \[ C = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (7 \mid n) \} \]

- How big are these sets? We use the floor function
  \[ |A| = \left\lfloor \frac{300}{3} \right\rfloor = 100 \]
  \[ |B| = \left\lfloor \frac{300}{5} \right\rfloor = 60 \]
  \[ |C| = \left\lfloor \frac{300}{7} \right\rfloor = 42 \]
Application of PIE: Example A (2)

• How many integers between 1 and 300 (inclusive) are divisible by at least one of 3, 5, 7?

Answer: \(|A \cup B \cup C|

• By the principle of inclusion-exclusion

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

• How big are these sets? We use the floor function

\(|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100 \quad |A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20
\]
\(|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60 \quad |A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 100
\]
\(|C| = \left\lfloor \frac{300}{7} \right\rfloor = 42 \quad |B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8
\]
\(|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2
\]

• Therefore:

\[ |A \cup B \cup C| = 100 + 60 + 42 - (20+14+8) + 2 = 162\]
Application of PIE: Example A (3)

- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?
  Answer: \(|(A \cap B) \setminus C|\)
- By the definition of set-minus
  \[|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18\]

- Knowing that
  \[|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100\]
  \[|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20\]
  \[|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60\]
  \[|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 100\]
  \[|C| = \left\lfloor \frac{300}{7} \right\rfloor = 42\]
  \[|B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8\]
  \[|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2\]
Application of PIE: Example A (4)

• How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?

Answer: \(|B\setminus(A \cup C)| = |B| - |B \cap (A \cup C)|

• Distributing B over the intersection

\(|B \cap (A \cup C)| = |(B \cap A) \cup (B \cap C)|
= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|
= |B \cap A| + |B \cap C| - |B \cap A \cap C|
= 20 + 8 − 2 = 26

• Knowing that

\(|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100\)
\(|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20\)
\(|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60\)
\(|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 14\)
\(|C| = \left\lfloor \frac{300}{7} \right\rfloor = 42\)
\(|B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8\)
\(|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2\)
Application of PIE: #Surjections

(Section 8.6)

• The principle of inclusion-exclusion can be used to count the number of onto (surjective) functions

• **Theorem:** Let A, B be non-empty sets of cardinality m, n with m \( \geq \) n. Then there are

\[
\begin{align*}
&n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m \\
&\text{i.e. } \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m \text{ onto functions } f : A \rightarrow B.
\end{align*}
\]

*See textbook, Section 8.6 page 561*
#Surjections: Example

- How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?
- This problem can be modeled as follows:
  - Let $A$ be the set of candies, $|A|=6$
  - Let $B$ be the set of children, $|B|=3$
  - The problem becomes “find the number of surjective mappings from $A$ to $B$” (because each child must receive at least one candy)
- Thus the number of ways is thus ($m=6$, $n=3$)

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$
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Pigeonhole Principle (1)

• If there are more pigeons than there are roots (pigeonholes), for at least one pigeonhole, more than one pigeon must be in it

• **Theorem**: If $k+1$ or more objects are placed in $k$ boxes, then there is at least one box containing two or more objects

• This principal is a fundamental tool of elementary discrete mathematics.

• It is also known as the [Dirichlet Drawer Principle](#) or [Dirichlet Box Principle](#)
Pigeonhole Principle (2)

• It is seemingly simple but very powerful
• The difficulty comes in where and how to apply it
• Some simple applications in Computer Science
  – Calculating the probability of hash functions having a collision
  – Proving that there can be no lossless compression algorithm compressing all files to within a certain ration

• Lemma: For two finite sets A, B there exists a bijection $f:A \rightarrow B$ if and only if $|A| = |B|$
Generalized Pigeonhole Principle (1)

• **Theorem**: If $N$ objects are placed into $k$ boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

• **Example**: In any group of 367 or more people, at least two of them must have been born on the same date.
Generalized Pigeonhole Principle (2)

• A probabilistic generalization states that
  – if \( n \) objects are randomly put into \( m \) boxes
  – **with uniform probability**
  – (i.e., each object is placed in a given box with probability \( 1/m \))
  – then at least one box will hold more than one object with probability

\[
1 - \frac{m!}{(m - n)! m^n}
\]
Generalized Pigeonhole Principle: Example

• Among 10 people, what is the probability that two or more will have the same birthday?
  – Here \( n=10 \) and \( m=365 \) (ignoring leap years)
  – Thus, the probability that two will have the same birthday is
  \[
  1 - \frac{365!}{(365 - 10)!365^{10}} \approx 0.1169
  \]
  So, less than 12% probability
Pigeonhole Principle: Example A (1)

• Show that
  – in a room of n people with certain acquaintances,
  – some pair must have the same number of acquaintances

• Note that this is equivalent to showing that any symmetric, irreflexive relation on n elements must have two elements with the same number of relations

• Proof: by contradiction using the pigeonhole principle
• Assume, to the contrary, that every person has a different number of acquaintances: 0, 1, 2, …, n-1
• Note: no one can have n acquaintances because the relation is irreflexive).
• There are n possibilities, we have n people, we are not done 😞
Pigeonhole Principle: Example A (2)

• There are n possibilities, we have n people, we are not done 😞
• Remember: acquaintanceship is a symmetric, irreflexive relation
• In particular
  – Some person knows 0 people
  – While another knows n-1 people, meaning knows the person who knows 0 people
• This situation is impossible. Contradiction! 😊
• So we do not have n (10) possibilities, but less
• Thus by the pigeonhole principle (10 people and 9 possibilities) at least two people have to the same number of acquaintances
Pigeonhole Principle: Example B

• **Example**: Say, 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats.

• **Show that**
  – One of the buses will have 14 empty seats
  – One of the buses will carry at least 67 passengers

• **One of the buses will have 14 empty seats**
  – Total number of seats is $80 \times 30 = 2400$
  – Total number of empty seats is $2400 - 2000 = 400$
  – By the pigeonhole principle: 400 empty seats in 30 buses, one must have $\lceil \frac{400}{30} \rceil = 14$ empty seats

• **One of the buses will carry at least 67 passengers**
  – By the pigeonhole principle: 2000 passengers in 30 buses, one must have $\lceil \frac{2000}{30} \rceil = 67$ passengers
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Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of \( r \) elements of a set of \( n \) elements is called an \( r \)-permutation
- **Theorem**: The number of \( r \) permutations of a set of \( n \) distinct elements is

\[
P(n, r) = \prod_{i=0}^{r-1} (n - i) = n(n - 1)(n - 2) \cdots (n - r + 1)
\]

- It follows that

\[
P(n, r) = \frac{n!}{(n - r)!}
\]

- In particular

\[
P(n, n) = n!
\]

- Note here that the **order is important**. It is necessary to distinguish when the order matters and it does not
Application of PIE and Permutations: Derangements (I)  
(Section 8.6)

• Consider the hat-check problem
  – Given
    • An employee checks hats from n customers
    • However, s/he forgets to tag them
    • When customers check out their hats, they are given one at random
  – Question
    • What is the probability that no one will get their hat back?
Application of PIE and Permutations: Derangements (II)

- The hat-check problem can be modeled using derangements: permutations of objects such that no element is in its original position.
  - Example: 21453 is a derangement of 12345 but 21543 is not.

- The number of derangements of a set with \( n \) elements is

\[
D_n = n! \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!} \right]
\]

- Thus, the answer to the hat-check problem is \( \frac{D_n}{n!} \).

- Note that

\[
e^{-1} = \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!} \right]
\]

- Thus, the probability of the hat-check problem converges

\[
\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1} \approx 0.368
\]

See textbook, Section 8.6 page 562
Permutations: Example A

• How many pairs of dance partners can be selected from a group of 12 women and 20 men?
  – The first woman can partner with any of the 20 men, the second with any of the remaining 19, etc.
  – To partner all 12 women, we have
    \[ P(20,12) = \frac{20!}{8!} = 9.10.11...20 \]
Permutations: Example B

• In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?
  – The number of ways is $P(24,10)$
  – Since we can choose either a or z to come first, then there are $2P(24,10)$ arrangements of the 12-letter block
  – For the remaining 14 letters, there are $P(15,15)=15!$ possible arrangements
  – In all there are $2P(24,10).15!$ arrangements
Permutations: Example C (1)

- How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern \(bge\) nor \(eaf\)?
  - The total number of permutations is \(P(7,7)=7!\)
  - If we fix the pattern \(bge\), then we consider it as a single block. Thus, the number of permutations with this pattern is \(P(5,5)=5!\)
  - Fixing the pattern \(eaf\), we have the same number: 5!
  - Thus, we have \((7! – 2.5!)\). Is this correct?
  - No! we have subtracted too many permutations: ones containing both \(eaf\) and \(bfe\).
Permutations: Example C (2)

- There are two cases: (1) \(eaf\) comes first, (2) \(bge\) comes first

- Are there any cases where \(eaf\) comes before \(bge\)?

- No! The letter \(e\) cannot be used twice

- If \(bge\) comes first, then the pattern must be \(bgeaf\), so we have 3 blocks or 3! arrangements

- Altogether, we have

\[7! - 2 \times (5!) + 3! = 4806\]
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Combinations (1)

- Whereas permutations consider order, combinations are used when order does not matter.
- **Definition**: A k-combination of elements of a set is an unordered selection of k elements from the set.
  
  (A combination is imply a subset of cardinality k)
Combinations (2)

- **Theorem**: The number of $k$-combinations of a set of cardinality $n$ with $0 \leq k \leq n$ is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is read ‘$n$ choose $k$’. $\binom{n}{k}$
Combinations (3)

• A useful fact about combinations is that they are symmetric

$$\binom{n}{1} = \binom{n}{n-1} \quad \binom{n}{2} = \binom{n}{n-2} \quad \binom{n}{3} = \binom{n}{n-3}$$

• Corollary: Let $n$, $k$ be nonnegative integers with $k \leq n$, then

$$\binom{n}{k} = \binom{n}{n-k}$$
Combinations: Example A

• In the Powerball lottery, you pick
  – Five numbers between 1 and 55 and
  – A single ‘powerball’ number between 1 and 42
   How many possible plays are there?

• Here order does not matter
  – The number of ways of choosing 5 numbers is \( \binom{55}{5} \)
  – There are 42 possible ways to choose the powerball
  – The two events are not mutually exclusive:
    \[ \frac{1}{42 \binom{55}{5}} < 0.0000000006845 \]
Combinatorics: Example B

- In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?
  - The number of ways of choosing 3 heads out of 10 coin tosses is \( \binom{10}{3} \)
  - It is the same as choosing 7 tails out of 10 coin tosses \( \binom{10}{7} = \binom{10}{3} = 120 \)
  - ... which illustrates the corollary \( \binom{n}{k} = \binom{n}{n-k} \)
Combinatorics: Example C

- How many committees of 5 people can be chosen from 20 men and 12 women
  - If exactly 3 men must be on each committee?
  - If at least 4 women must be on each committee?

- **If exactly three men must be on each committee?**
  - We must choose 3 men and 2 women. The choices are not mutually exclusive, we use the product rule
    \[
    \binom{20}{3} \cdot \binom{12}{2}
    \]

- **If at least 4 women must be on each committee?**
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. These choices are mutually exclusive, we use the addition rule:
    \[
    \binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692
    \]
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Binomial Coefficients (1)

- The number of \( r \)-combinations \( \binom{n}{r} \) is also called the \textbf{binomial coefficient}.

- The binomial coefficients are the coefficients in the expansion of the expression, (multivariate polynomial),

\[
(x+y)^n
\]

- A binomial is a sum of two terms.
Binomial Coefficients (2)

• **Theorem:** Binomial Theorem

Let $x$, $y$, be variables and let $n$ be a nonnegative integer. Then

\[(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j\]

Expanding the summation we have

\[(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n\]

Example

\[(x + y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3\]
Binomial Coefficients: Example

- What is the coefficient of the term $x^8y^{12}$ in the expansion of $(3x+4y)^{20}$?
  - By the binomial theorem, we have
    \[(3x + 4y)^{20} = \sum_{j=0}^{20} \binom{20}{j} (3x)^{n-j} (4y)^j\]
  - When $j=12$, we have
    \[\binom{20}{12} (3x)^8 (4y)^{12}\]
  - The coefficient is
    \[\binom{20}{12} 3^8 4^{12} = \frac{20!}{12!8!} 3^8 4^{12} = 13866187326750720\]
Binomial Coefficients (3)

- Many useful identities and facts come from the Binomial Theorem

- **Corollary:**

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad n \geq 1
\]

\[
\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n
\]

*Equalities are based on* \((1+1)^n=2^n\), \((-1)+1)^n=0^n\), \((1+2)^n=3^n\)
Binomial Coefficients (4)

• **Theorem:** Vandermonde’s Identity
  Let \( m, n, r \) be nonnegative integers with \( r \) not exceeding either \( m \) or \( n \). Then
  \[
  \binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}
  \]

• **Corollary:** If \( n \) is a nonnegative integer then
  \[
  \binom{2n}{n} = \sum_{k=0}^{r} \binom{n}{k}^2
  \]

• **Corollary:** Let \( n,r \) be nonnegative integers, \( r \leq n \), then
  \[
  \binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}
  \]
Binomial Coefficients: Pascal’s Identity & Triangle

• The following is known as Pascal’s identity which gives a useful identity for efficiently computing binomial coefficients

• **Theorem**: Pascal’s Identity

Let $n,k \in \mathbb{Z}^+$ with $n \geq k$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal’s Identity forms the basis of a geometric object known as Pascal’s Triangle
Pascal’s Triangle

\[
\begin{array}{cccccccc}
0 & 0 \\
1 & 0 & 1 \\
2 & 0 & 2 & 1 \\
3 & 0 & 3 & 3 & 1 \\
4 & 0 & 4 & 6 & 4 & 1 \\
5 & 0 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]
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Generalized Combinations & Permutations (1)

• Sometimes, we are interested in permutations and combinations in which repetitions are allowed

• **Theorem**: The number of $r$-permutations of a set of $n$ objects with repetition allowed is $n^r$

  ...which is easily obtained by the product rule

• **Theorem**: There are

  $${\binom{n + r - 1}{r}}$$

  $r$-combinations from a set with $n$ elements when repetition of elements is allowed
Generalized Combinations & Permutations: Example

• There are 30 varieties of donuts from which we wish to buy a dozen. How many possible ways to place your order are there?
• Here, $n=30$ and we wish to choose $r=12$.
• Order does not matter and repetitions are possible
• We apply the previous theorem
• The number of possible orders is

\[
\binom{n + r - 1}{r} = \binom{30 + 12 - 1}{12} = \binom{17}{12}
\]
Generalized Combinations & Permutations (2)

- **Theorem:** The number of different permutations of n objects where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) of type 2, and \( n_k \) of type k is

\[
\frac{n!}{n_1! n_2! \cdots n_k!}
\]

An equivalent ways of interpreting this theorem is the number of ways to
- distribute \( n \) distinguishable objects
- into \( k \) distinguishable boxes
- so that \( n_i \) objects are place into box \( i \) for \( i=1,2,3,...,k \)
Example

• How many permutations of the word Mississipi are there?

• ‘Mississipi’ has
  – 4 distinct letters: m,i,s,p
  – with 1,4,4,2 occurrences respectively
  – Therefore, the number of permutations is

\[
\frac{11!}{1!4!4!2!}
\]
Outline

• Introduction
• Counting:
  – Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  – Application of PIE: Number of onto functions
• Pigeonhole principle
  – Generalized, probabilistic forms
• Permutations
• Combinations
• Binomial Coefficients
• Generalizations
  – Combinations with repetitions, permutations with indistinguishable objects

• Algorithms
  – Generating combinations (1), permutations (2)
• More Examples
Algorithms

• In general, it is inefficient to solve a problem by considering all permutation or combinations since there are exponential (worst, factorial!) numbers of such arrangements.

• Nevertheless, for many problems, no better approach is known.

• When exact solutions are needed, backtracking algorithms are used to exhaustively enumerate all arrangements.
Algorithms: Example

• **Traveling Salesperson Problem (TSP)**
  Consider a salesman that must visit \( n \) different cities. He wishes to visit them in an order such that his overall distance travelled is minimized.

• This problem is one of hundred of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that no efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP.)

• The only way of solving this problem **exactly** is to try all possible \( n! \) routes.

• We give several algorithms for generating these combinatorial objects.
Generating Combinations (1)

• Recall that combinations are simply all possible subsets of size $r$. For our purposes, we will consider generating subsets of 
  \{1,2,3,...,n\}

• The algorithm works as follows
  – Start with \{1,...,r\}
  – Assume that we have $a_1a_2...a_r$, we want the next combination
  – Locate the last element $a_i$ such that $a_i \neq n-r-1$
  – Replace $a_i$ with $a_i+1$
  – Replace $a_j$ with $a_i+j-1$ for $j=i+1, i+2,...,r$
Generating Combinations (2)

**Next r-Combinations**

*Input:* A set of $n$ elements and an $r$-combination $a_1, a_2, \ldots, a_r$

*Output:* The next $r$-combination

1. $i \leftarrow r$
2. **While** $a_i = n-r+i$ **Do**
3. \hspace{1cm} $i \leftarrow i - 1$
4. **End**
5. $a_i \leftarrow a_i + 1$
6. **For** $j \leftarrow (i+1)$ **to** $r$ **Do**
7. \hspace{1cm} $a_j \leftarrow a_i + j - i$
8. **End**
Generating Combinations: Example

• Find the next 3-combination of the set \{1,2,3,4,5\} after \{1,4,5\}
• Here \(a_1=1, a_2=4, a_3=5, n=5, r=3\)
• The last \(i\) such that \(a_i \neq 5-3+i\) is 1
• Thus, we set
  \[
  a_1 = a_1 + 1 = 2 \\
  a_2 = a_1 + 2 - 1 = 3 \\
  a_3 = a_1 + 3 - 1 = 4
  \]
  Thus, the next \(r\)-combinations is \{2,3,4\}
Generating Permutations

- The textbook gives an algorithm to generate permutations in lexicographic order. Essentially, the algorithm works as follows. Given a permutation
  - Choose the left-most pair $a_j, a_{j+1}$ where $a_j < a_{j+1}$
  - Choose the least items to the right of $a_j$ greater than $a_j$
  - Swap this item and $a_j$
  - Arrange the remaining (to the right) items in order
Next Permutation (lexicographic order)

**Input**: A set of $n$ elements and an $r$-permutation, $a_1 \cdots a_r$.

**Output**: The next $r$-permutation.

1. $j = n - 1$
2. **WHILE** $a_j > a_{j+1}$ **DO**
   3. $j = j - 1$
4. **END**
   // $j$ is the largest subscript with $a_j < a_{j+1}$
5. $k = n$
6. **WHILE** $a_j > a_k$ **DO**
   7. $k = k - 1$
8. **END**
   // $a_k$ is the smallest integer greater than $a_j$ to the right of $a_j$
9. **swap**($a_j, a_k$)
10. $r = n$
11. $s = j + 1$
12. **WHILE** $r > s$ **DO**
13. **swap**($a_r, a_s$)
14. $r = r - 1$
15. $s = s + 1$
16. **END**
Generating Permutations (2)

- Often there is no reason to generate permutations in lexicographic order. Moreover even though generating permutations is inefficient in itself, lexicographic order induces even more work.
- An alternate method is to fix an element, then recursively permute the n-1 remaining elements.
- The Johnson-Trotter algorithm has the following attractive properties. Not in your textbook, not on the exam, just for your reference/culture:
  - It is bottom up (non-recursive)
  - It induces a minimal-change between each permutation.
Johnson-Trotter Algorithm

• We associate a direction to each element, for example

\[
\begin{array}{cccc}
3 & 2 & 4 & 1 \\
\end{array}
\]

• A component is mobile if its direction points to an adjacent component that is smaller than itself.

• Here 3 and 4 are mobile, 1 and 2 are not
Algorithm: Johnson Trotter

**Input**: An integer $n$.

**Output**: All possible permutations of $\left\langle 1, 2, \ldots, n \right\rangle$.

1. $\pi = [1, 2, \ldots, n]$
2. While There exists a mobile integer $k \in \pi$ do
   3. $k =$ largest mobile integer
   4. swap $k$ and the adjacent integer $k$ points to
   5. reverse direction of all integers $> k$
   6. Output $\pi$
7. END
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Example A

• How many bit strings of length 4 are there such that 11 never appear as a substring

• We can represent the set of strings graphically using a diagram tree (see textbook pages 395)

```
0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110
1111
```
Example: Counting Functions (1)

• Let $S, T$ be sets such that $|S| = n$, $|T| = m$.
  – How many functions are there mapping $f : S \rightarrow T$?
  – How many of these functions are one-to-one (injective)?

• A function simply maps each $s_i$ to one $t_j$, thus for each $n$ we can choose to send it to any of the elements in $T$

• Each of these is an independent event, so we apply the multiplication rule:

• If we wish $f$ to be injective, we must have $n \leq m$, otherwise the answer is obviously 0
Example: Counting Functions (2)

- Now each $s_i$ must be mapped to a unique element in $T$.
  - For $s_1$, we have $m$ choices.
  - However, once we have made a mapping, say $s_j$, we cannot map subsequent elements to $t_j$ again.
  - In particular, for the second element, $s_2$, we now have $m-1$ choices, for $s_3$, $m-2$ choices, etc.
    $$m \cdot (m-1) \cdot (m-2) \cdot \ldots \cdot (m-(n-2)) \cdot (m-(n-1))$$

- An alternative way of thinking is using the choose operator: we need to choose $n$ elements from a set of size $m$ for our mapping.
  $$\binom{m}{n} = \frac{m!}{(m-n)!n!}$$

- Once we have chosen this set, we now consider all permutations of the mapping, that is $n!$ different mappings for this set. Thus, the number of such mapping is
  $$\frac{m!}{(m-n)!n!} \cdot n! = \frac{m!}{(m-n)!}$$
Another Example: Counting Functions

• Let $S=\{1,2,3\}$, $T=\{a,b\}$.
  – How many onto (surjective) mappings are there from $S \rightarrow T$?
  – How many onto-to-one injective functions are there from $T \rightarrow S$?

• See Theorem 1, page 561
Example: Sets

- How many $k$ integers $1 \leq k \leq 100$ are divisible by 2 or 3?
- Let
  - $A = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (2 \mid n)\}$
  - $B = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (3 \mid n)\}$
- Clearly, $|A| = \lfloor 100/2 \rfloor = 50$, $|B| = \lfloor 100/3 \rfloor = 33$
- Do we have $|A \cup B| = 83$? No!
- We have over counted the integers divisible by 6
  - Let $C = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (6 \mid n)\}$, $|C| = \lfloor 100/6 \rfloor = 16$
- So $|A \cup B| = (50+33) - 16 = 67$
Summary

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