Asymptotics

Section 3.2 of Rosen
Spring 2017
CSCE 235H Introduction to Discrete Structures (Honors)
Course web-page: cse.unl.edu/~cse235h
Questions: Piazza
Outline

• Introduction
• Asymptotic
  – Definitions (Big O, Omega, Theta), properties
• Proof techniques
  – 3 examples, trick for polynomials of degree 2,
  – Limit method (l’Hôpital Rule), 2 examples
• Limit Properties
• Efficiency classes
• Conclusions
Introduction (1)

• We are interested only in the Order of Growth of an algorithm’s complexity
• How well does the algorithm perform as the size of the input grows: $n \rightarrow \infty$
• We have seen how to mathematically evaluate the cost functions of algorithms with respect to
  – their input size $n$ and
  – their elementary operations
• However, it suffices to simply measure a cost function’s asymptotic behavior
Introduction (2): Magnitude Graph

FIGURE 3  A Display of the Growth of Functions Commonly Used in Big-O Estimates.
Introduction (3)

• In practice, specific hardware, implementation, languages, etc. greatly affect how the algorithm behave

• Our goal is to study and analyze the behavior of algorithms in and of themselves, independently of such factors

• For example
  – An algorithm that executes its elementary operation $10n$ times is better than one that executes it $0.005n^2$ times
  – Also, algorithms that have running time $n^2$ and $2000n^2$ are considered asymptotically equivalent
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• Introduction
• **Asymptotic**
  – Definitions (Big-O, Omega, Theta), properties
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Big-O Definition

• **Definition**: Let $f$ and $g$ be two functions $f, g: \mathbb{N} \rightarrow \mathbb{R}^+$. We say that

$$f(n) \in O(g(n))$$

(read: $f$ is Big-O of $g$) if there exists a constant $c \in \mathbb{R}^+$ and an $n_0 \in \mathbb{N}$ such that for every integer $n \geq n_0$ we have

$$f(n) \leq cg(n)$$

• Big-O is actually Omicron, but it suffices to write “O”

  Intuition: $f$ is asymptotically less than or equal to $g$

• Big-O gives an asymptotic upper bound $\mathcal{O}$
Big-Omega Definition

- **Definition**: Let $f$ and $g$ be two functions $f,g: N \rightarrow R^+$. We say that

  \[ f(n) \in \Omega(g(n)) \]

  (read: $f$ is Big-Omega of $g$) if there exists a constant $c \in R^+$ and an $n_0 \in N$ such that for every integer $n \geq n_0$ we have

  \[ f(n) \geq cg(n) \]

- **Intuition**: $f$ is asymptotically greater than or equal to $g$

- **Big-Omega gives an asymptotic lower bound**
Big-Theta Definition

- **Definition**: Let \( f \) and \( g \) be two functions \( f, g: \mathbb{N} \rightarrow \mathbb{R}^+ \). We say that

\[
f(n) \in \Theta(g(n))
\]

(read: \( f \) is Big-Omega of \( g \)) if there exists a constant \( c_1, c_2 \in \mathbb{R}^+ \) and an \( n_0 \in \mathbb{N} \) such that for every integer \( n \geq n_0 \) we have

\[
c_1 g(n) \leq f(n) \leq c_2 g(n)
\]

- Intuition: \( f \) is asymptotically equal to \( g \)
- \( f \) is bounded above and below by \( g \)
- Big-Theta gives an asymptotic equivalence \( \Theta() \)
Asymptotic Properties (1)

- **Theorem**: For \( f_1(n) \in O(g_1(n)) \) and \( f_2(n) \in O(g_2(n)) \), we have
  \[
  f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})
  \]

- This property implies that we can ignore lower order terms. In particular, for any polynomial with degree \( k \) such as
  \[
  p(n) = an^k + bn^{k-1} + cn^{k-2} + ...,
  \]
  \[
  p(n) \in O(n^k)
  \]
  More accurately, \( p(n) \in \Theta(n^k) \)

- In addition, this theorem gives us a justification for ignoring constant coefficients. That is for any function \( f(n) \) and a positive constant \( c \)
  \[
  cf(n) \in \Theta(f(n))
  \]
Asymptotic Properties (2)

• Some obvious properties also follow from the definitions

• **Corollary**: For positive functions $f(n)$ and $g(n)$ the following hold:
  - $f(n) \in \Theta(g(n)) \iff f(n) \in O(g(n)) \land f(n) \in \Omega(g(n))$
  - $f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$

The proof is obvious and left as an exercise
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Asymptotic Proof Techniques

• Proving an asymptotic relationship between two given function $f(n)$ and $g(n)$ can be done intuitively for most of the functions you will encounter; all polynomials for example
• However, this does not suffice as a formal proof
• To prove a relationship of the form $f(n) \in \Delta(g(n))$, where $\Delta$ is $O$, $\Omega$, or $\Theta$, can be done using the definitions, that is
  – Find a value for $c$ (or $c_1$ and $c_2$)
  – Find a value for $n_0$

(But the above is not the only way.)
Asymptotic Proof Techniques: Example A

Example: Let $f(n) = 21n^2 + n$ and $g(n) = n^3$

- Our intuition should tell us that $f(n) \in O(g(n))$
- Simply using the definition confirms this:
  \[ 21n^2 + n \leq cn^3 \]
  holds for say $c=3$ and for all $n \geq n_0 = 8$
- So we found a pair $c=3$ and $n_0=8$ that satisfy the conditions required by the definition QED
- In fact, an infinite number of pairs can satisfy this equation
Asymptotic Proof Techniques: Example B (1)

- **Example**: Let \( f(n) = n^2 + n \) and \( g(n) = n^3 \). Find a tight bound of the form
  \[
  f(n) \in \Delta(g(n))
  \]
- Our intuition tells us that \( f(n) \in O(g(n)) \)
- Let’s prove it formally
Example B: Proof

• If $n \geq 1$ it is clear that
  1. $n \leq n^3$ and
  2. $n^2 \leq n^3$

• Therefore, we have, as 1. and 2.:
  
  \[ n^2 + n \leq n^3 + n^3 = 2n^3 \]

• Thus, for $n_0 = 1$ and $c = 2$, by the definition of Big-O we have that $f(n) = n^2 + n \in O(g(n^3))$
Asymptotic Proof Techniques: Example C (1)

- **Example**: Let $f(n) = n^3 + 4n^2$ and $g(n) = n^2$. Find a tight bound of the form
  
  $$f(n) \in \Delta(g(n))$$

- Here, our intuition tells us that $f(n) \in \Omega(g(n))$

- Let’s prove it formally
Example C: Proof

• For $n \geq 1$, we have $n^2 \leq n^3$

• For $n \geq 0$, we have $n^3 \leq n^3 + 4n^2$

• Thus $n \geq 1$, we have $n^2 \leq n^3 \leq n^3 + 4n^2$

• Thus, by the definition of Big-$\Omega$, for $n_0 = 1$ and $c = 1$ we have that $f(n) = n^3 + 4n^2 \in \Omega(g(n^2))$
Asymptotic Proof Techniques: Trick for polynomials of degree 2

- If you have a polynomial of degree 2 such as
  \[ an^2 + bn + c \]
  you can prove that it is \( \Theta(n^2) \) using the following values

1.  \( c_1 = a/4 \)
2.  \( c_2 = 7a/4 \)
3.  \( n_0 = 2 \max(|b|/a, \sqrt{|c|}/a) \)
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Limit Method: Motivation

- Now try this one:
  \[ f(n) = n^{50} + 12n^3 \log^4 n - 1243n^{12} + 245n^6 \log n + 12 \log^3 n - \log n \]
  \[ g(n) = 12n^{50} + 24 \log^{14} n^{43} - \log n/n^5 + 12 \]

- Using the formal definitions can be very tedious especially one has very complex functions

- It is much better to use the Limit Method, which uses concepts from Calculus
Limit Method: The Process

• Say we have functions \( f(n) \) and \( g(n) \). We set up a limit quotient between \( f \) and \( g \) as follows

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 
0 & \text{Then } f(n) \in O(g(n)) \\
\infty & \text{Then } f(n) \in \Omega(g(n)) \\
c > 0 & \text{Then } f(n) \in \Theta(g(n))
\end{cases}
\]

• The above can be proven using calculus, but for our purposes, the limit method is sufficient for showing asymptotic inclusions

• Always try to look for algebraic simplifications first

• If \( f \) and \( g \) both diverge or converge on zero or infinity, then you need to apply the l’Hôpital Rule
(Guillaume de) L’Hôpital Rule

• Theorem (L’Hôpital Rule):
  – Let \( f \) and \( g \) be two functions,
  – if the limit between the quotient \( f(n)/g(n) \) exists,
  – Then, it is equal to the limit of the derivative of
    the numerator and the denominator

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}
\]
Useful Identities & Derivatives

• Some useful derivatives that you should memorize
  – \((n^k)' = k \, n^{k-1}\)
  – \((\log_b (n))' = 1/(n \, \ln(b))\)
  – \((f_1(n)f_2(n))' = f_1' (n)f_2(n)+f_1(n)f_2' (n) \quad (product \ rule)\)
  – \((\log_b(f(n))' = f' (n)/(f(n) \, \ln b)\)
  – \((c^n)' = \ln(c)c^n \quad \leftarrow \text{careful!}\)

• Log identities
  – Change of base formula: \(\log_b(n) = \log_c(n)/\log_c(b)\)
  – \(\log(n^k) = k \log(n)\)
  – \(\log(ab) = \log(a) + \log(b)\)
L’Hôpital Rule: Justification (1)

• Why do we have to use L’Hôpital’s Rule?
• Consider the following function
  \[ f(x) = \frac{\sin x}{x} \]
  • Clearly \( \sin 0 = 0 \). So you may say that when \( x \to 0 \), \( f(x) \to 0 \)
  • However, the denominator is also \( \to 0 \), so you may say that \( f(x) \to \infty \)
• Both are wrong
L’Hôpital Rule: Justification (2)

- Observe the graph of $f(x) = (\sin x)/x = \text{sinc } x$
L’Hôpital Rule: Justification (3)

- Clearly, though \( f(x) \) is undefined at \( x=0 \), the limit still exists.
- Applying the L’Hôpital Rule gives us the correct answer:
  \[
  \lim_{x \to 0} \left( \frac{\sin x}{x} \right) = \lim_{x \to 0} \left( \frac{\sin x}{x}' / x' \right) = \frac{\cos x}{1} = 1
  \]
Limit Method: Example 1

• Example: Let $f(n) = 2^n$, $g(n) = 3^n$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$

• What is your intuition in this case? Which function grows quicker?
Limit Method: Example 1—Proof A

• Proof using limits
• We set up our limit:
  \[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{2^n}{3^n} \]
• Using L’Hôpital Rule gets you no where
  \[ \lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} \frac{(2^n)'}{(3^n)'} = \lim_{n \to \infty} \frac{(\ln 2)(2^n)}{(\ln 3)(3^n)} \]
• Both the numerator and denominator still diverge. We’ll have to use an algebraic simplification
Limit Method: Example 1—Proof B

• Using algebra

\[ \lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n \]

• Now we use the following Theorem w/o proof

\[ \lim_{n \to \infty} \alpha^n = \begin{cases} 
0 & \text{if } \alpha < 1 \\
1 & \text{if } \alpha = 1 \\
\infty & \text{if } \alpha > 1 
\end{cases} \]

• Therefore we conclude that the \( \lim_{n \to \infty} \left(\frac{2}{3}\right)^n \) converges to zero thus \( 2^n \in O(3^n) \)
Limit Method: Example 2 (1)

• Example: Let $f(n) = \log_2 n$, $g(n) = \log_3 n^2$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$.

• What is your intuition in this case?
Limit Method: Example 2 (2)

• We prove using limits
• We set up out limit
  \[ \lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{\log_2 n}{\log_3 n^2} = \lim_{n \to \infty} \frac{\log_2 n}{2\log_3 n} \]
• Here we use the change of base formula for logarithms: \( \log_x n = \frac{\log_y n}{\log_y x} \)
• Thus: \( \log_3 n = \frac{\log_2 n}{\log_2 3} \)
Limit Method: Example 2 (3)

• Computing our limit:

\[ \lim_{n \to \infty} \frac{\log_2 n}{2 \log_3 n} = \lim_{n \to \infty} \frac{\log_2 n \log_2 3}{2 \log_2 n} \]

\[ = \lim_{n \to \infty} \frac{\log_2 3}{2} \]

\[ = \frac{\log_2 3}{2} \approx 0.7924, \text{ which is a positive constant} \]

• So we conclude that \( f(n) \in \Theta(g(n)) \)
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Limit Properties

• A useful property of limits is that the composition of functions is preserved

• Lemma: For the composition \( \circ \) of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

\[
\lim_{n \to \infty} f_1(n) \circ \lim_{n \to \infty} f_2(n) = \lim_{n \to \infty} (f_1(n) \circ f_2(n))
\]
## Complexity of Algorithms—Table 1, page 226

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>$O(\log (n))$</td>
</tr>
<tr>
<td>Linear</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Polylogarithmic</td>
<td>$O(\log^k (n))$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Cubic</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Polynominal</td>
<td>$O(n^k)$ for any $k&gt;0$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$O(k^n)$, where $k&gt;1$</td>
</tr>
<tr>
<td>Factorial</td>
<td>$O(n!)$</td>
</tr>
</tbody>
</table>
Conclusions

• Evaluating asymptotics is easy, but remember:
  – *Always* look for algebraic simplifications
  – You must *always* give a rigorous proof
  – Using the limit method is (almost) always the best
  – Use L’Hôpital Rule if need be
  – Give as simple *and tight* expressions as possible
Summary

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