Recitation 10

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- Induction: Example using triominoes for $2^n \times 2^n$ checkerboard missing one corner, see page 326.
- Problem 5.1.5: Using induction, prove:

$$\forall n \geq 0 \; 1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

  We prove the property using mathematical induction:

  1. Let's state the property to prove:

  $$P(n) : 1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

  2. We take $n_0 = 0$ and prove $P(0)$:

  $$P(0) = 1 = \frac{(1)(1)(3)}{3}$$

  We show that P(0) holds: $1 = 1.\frac{1 \cdot 1 \cdot 3}{3} = 1$. Therefore, P(0) is true.

  3. What is the inductive hypothesis?

  $$P(k) : 1^2 + 3^2 + \ldots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

  4. Assuming the base case and the inductive hypothesis, we want to prove:

  $$P(k+1) : 1^2 + 3^2 + \ldots + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

  5. We start from $1^2 + 3^2 + \ldots + (2k+1)^2 + (2k+3)^2$.
  $1^2 + 3^2 + \ldots + (2k+1)^2 + (2k+3)^2$
  $= (1^2 + 3^2 + \ldots + (2k+1)^2) + (2k+3)^2$
  $= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$ *using the inductive hypothesis*
  $= (2k+3) \cdot (\frac{(k+1)(2k+1)+6k+9}{3})$
  $= (2k+3) \cdot (\frac{2k^2+9k+10}{3})$
  $= \frac{(k+2)(2k+3)(2k+5)}{3}$
  Thus, $P(k+1)$ holds.

Consequently, by the PMI, $\forall n \geq 0 \; 1^2 + 3^2 + \ldots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

- Now, prove the following: $3 \mid 2^{2n} - 1$ for $n \geq 1$.

  1. First, we state the property to prove: $P(n) : 3 \mid 2^{2n} - 1$.
  2. Base case is $n_0 = 1$. So $P(1) = 3 \mid 2^{2(1)} - 1$. Clearly, $2^{2(1)} - 1 = 4 - 1 = 3$, $3 \mid 3$, which is obvious. So, $P(1)$ holds.
  3. Next, we state the inductive hypothesis: $P(k) : 3 \mid 2^{2k} - 1$ and assume that $P(k)$ holds.
  4. Now, we have to prove that $P(k+1) : 3 \mid 2^{2k+2} - 1$ holds. We start from $2^{2k+2} - 1$.
     $2^{2k+2} - 1 = 4 \cdot 2^k - 1 = 4 \cdot (2^k - 1 + 1) - 1$.
     The inductive hypothesis gives that $3 \mid 2^{2k} - 1$ This, there is an integer t such that $2^{2k} - 1 = 3t$. Thus,
     $2^{2k+2} - 1 = 4 \cdot (3t + 1) - 1 = 12t + 4 - 1 = 12t + 3 = 3 \cdot (4t + 1)$.
     Thus, $2^{2k+2} - 1$ is a multiple of 3. Hence, $P(k+1)$ holds.

  Consequently, by the PMI, $\forall n \geq 1, \; 3 \mid 2^{2n} - 1$.

- A proof by strong induction.

  $$\text{Show that } \forall n \in \mathbb{N}, \; 12 \mid (n^4 - n^2).$$

  1. First, we state the property:
     $$P(n) : 12 \mid (n^4 - n^2)$$
  2. **Base Case:**
     (a) For $n = 1$: $1^4 - 1^2 = 0 = 12 \cdot 0$, so $P(1)$ is true.
     (b) For $n = 2$: $2^4 - 2^2 = 16 - 4 = 12 = 12 \cdot 1$, so $P(2)$ is true.
     (c) For $n = 3$: $3^4 - 3^2 = 81 - 9 = 72 = 12 \cdot 6$, so $P(3)$ is true.
     (d) For $n = 4$: $4^4 - 4^2 = 256 - 16 = 240 = 12 \cdot 20$, so $P(4)$ is true.
     (e) For $n = 5$: $5^4 - 5^2 = 625 - 25 = 600 = 12 \cdot 50$, so $P(5)$ is true.
     (f) For $n = 6$: $6^4 - 6^2 = 1296 - 36 = 1260 = 12 \cdot 105$, so $P(6)$ is true.
  3. **Strong Inductive Hypothesis** Let $k \geq 6 \in \mathbb{N}$ and assume that $12 \mid (m^4 - m^2)$ for $1 \leq m < k$ where $m \in \mathbb{N}$.
  4. We need to estalish $P(k)$.
     Consider $i = k - 5$. Given that $i < k$, we can assume that $P(i)$ holds.
     Clearly $i + 6 = (k - 5) + 6 = k + 1$.
     Let's compute: $(i+6)^4 - (i+6)^2$
     $$\begin{aligned} &= (i^4 + 24i^3 + 180i^2 + 864i + 1296) - (i^2 + 12i + 36) \\ &= (i^4 - i^2) + 24i^3 + 180i^2 + 852i + 1260 \\ &= (i^4 - i^2) + 12 \cdot (2i^3 + 15i^2 + 71i + 105) \end{aligned}$$

Given that $P(i)$ holds, we have $i^4 - i^2 = 12 \cdot t$.
Thus, $(i+6)^4 - (i+6)^2$

$$= 12 \cdot t + 12 \cdot (2i^3 + 15i^2 + 71i + 105)$$
$$= 12 \cdot (t + 2i^3 + 15i^2 + 71i + 105).$$

Hence, $(i+6)^4 - (i+6)^2$ is a multiple of 12 and $12 \mid (k+1)^4 - (k+1)^2$.

We can finally state that by the principle of strong induction, $\forall n \in \mathbb{N},\ 12 \mid (n^4 - n^2)$.