Recitation 7

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- Properties of a relation R on a set A:
  1. **Reflexive**: $(a, a) \in R$ for all $a \in A$
  2. **Symmetric**: $\forall a, b \in A$, $(a, b) \in R \to (b, a) \in R$
  3. **Antisymmetric**: $\forall a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
  4. **Transitive**: $\forall a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$
  5. **Irreflexive**: $\forall a \in A$, $(a, a) \notin R$
  6. **Asymmetric**: $\forall a, b \in A$ $(a, b) \in R$ then $(b, a) \notin R$
  7. **Equivalence Relation**: A relation that is *reflexive*, *symmetric*, and *transitive*.

- Problem 9.1.3 a: Determine whether the following relation over the set {1,2,3,4} is symmetric, antisymmetric, reflexive, and/or transitive,
  1. $R = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
  2. Is it reflexive? **No, there is no (4,4) element**
  3. Is it symmetric? **No, there are no (4,2), or (4,3) elements**
  4. Is it Antisymmetric? **No, (2,3) and (3,2) are elements**
  5. Is it Transitive? **Yes**

- How about 9.1.3 b: over {1,2,3,4}
  1. $S = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
  2. Is it reflexive? **Yes**
  3. Is it symmetric? **Yes**
  4. Is it Antisymmetric? **No, (2,1) and (1,2) are elements**
  5. Is it Transitive? **Yes**

- What is the relation $S \cup R$?

$$S \cup R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 4)\}$$

1. Antisymmetric?: **No, no (1,2) and (2,1)**
2. Symmetric? **No, no (4,2) element**
3. Reflexive? **Yes**
4. Transitive? **No** (1,2) and (2,4), but no (1,4)

- What is the relation $S \cap R$?

$$S \cap R = \{(2,2), (3,3)\}$$

1. Antisymmetric? **Yes**
2. Symmetric? **Yes**
3. Reflexive? **No**, missing $(1,1), (4,4)$. Neither reflexive nor irreflexive.
4. Transitive? **Yes**

- Represent S as a bit matrix: $M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Rosen 9.4.25(b). Use Algorithm 1 (Given on page 603) to compute the transitive closure of the relation $\{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$ on the set $\{1,2,3,4\}$.

  - We note the matrix of a relation R^x resulting from the composing the relation R with itself x times: M_{R^x}, alternatively: $M_R^{[x]}$.
  - We note the relations composition operator $\circ$ and the matrix product operator $\cdot$, alternatively, $\odot$.

$$M_R = M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = M_{R^1 \circ R^1} = M_{R^1} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R^3} = M_{R \circ R^2} = M_{R^2} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^4} = M_{R \circ R^3} = M_{R^3} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$
\begin{aligned}
M_{R^*} &= M_{R^1} \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

- Rosen 9.4.27(b)

$$
M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
$$

$$
W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
$$

  So the transitive closure looks like {(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4)}
  Is this transitive? **Yes**

- The following are relations on {1,2,3,0} are they equivalence relations?
  - {(0,0),(1,1),(2,2),(3,3)} **Yes** this one is fairly obvious, as everything just relates back to itself.
  - {(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)} **No**, missing (0,0) (I removed it this is not identical to 9.5 #1), so not reflexive.
  - {(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)} **Yes**

- Problem 9.5.47: {0}, {1,2},{3,4,5}
  - So here we'll have (a, b) iff a and b are in the same subset
  - So, $(0, 0)$ is an element.
  - (1,1),(1,2),(2,1),(2,2) are elements.
  - (3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5) are also elements.
  - Thus, our equivalence relation is

    $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$