Recitation 6

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- Problem 2.2:31: Show that for A and B subsets of some universal set U,

$$A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$$

$$\begin{aligned}
&A \subseteq B \Leftrightarrow && \\
&\quad \forall x, \quad x \in A \to x \in B && \text{Definitionof set inclusion} \\
&\quad \Leftrightarrow x \notin A \vee x \in B \Leftrightarrow && \text{Implication rule} \\
&\quad x \in B \vee x \notin A \Leftrightarrow && \text{Commutativity} \\
&\quad x \notin B \to x \notin A \Leftrightarrow && \text{Implication rule} \\
&\quad x \in \bar{B} \to x \in \bar{A} \Leftrightarrow && \text{Definition of set complement} \\
&\bar{B} \subseteq \bar{A} && \text{by definition of a set inclusion} \\
& && \text{QED}
\end{aligned}$$

- 2.2.37 c: Show that if A is a subset of universal set U

$$A \oplus U = \bar{A}$$

$$\begin{aligned}
\forall x, \quad & x \in A \oplus U \Leftrightarrow && \\
& ((x \in A) \vee (x \in U)) \wedge \neg((x \in A) \wedge (x \in U)) \Leftrightarrow && \text{definition of symmetric} \\
& && \text{difference } \oplus \text{ on page 137} \\
& ((x \in A \cup U)) \wedge \neg((x \in A \cap U)) \Leftrightarrow && \text{Definition set union, intersection} \\
& (x \in U) \wedge \neg(x \in A) \Leftrightarrow && \text{Domination, identity laws} \\
& \neg(x \in A) \wedge (x \in U) \Leftrightarrow && \text{Commutative law (logic)} \\
& (x \notin A) \wedge (x \in U) \Leftrightarrow && \text{Moving negation inward} \\
& x \in \bar{A} && \text{Definition of set absolute complement}
\end{aligned}$$

We showed that $\forall x,\ x \in A \oplus U \Leftrightarrow x \in \bar{A}$. Thus, $A \oplus U = \bar{A}$. $\square$

- Suppose that $A \cup B = \emptyset$, what can you conclude?
  Answer: we conclude that $(A = \emptyset) \wedge (B = \emptyset)$.

  We formally prove that:

$$A \cup B = \emptyset \Leftrightarrow (A = \emptyset) \wedge (B = \emptyset)$$

  First we consider prove the following statement:

$$A \cup B = \emptyset \to (A = \emptyset) \wedge (B = \emptyset)$$

The proof is by contradiction. We assume the antecedent and negate the conclusion.

(1)	$A \cup B = \emptyset$	given
(2)	$\neg((A = \emptyset) \wedge (B = \emptyset))$	negating the conclusion
(3)	$(A \neq \emptyset) \vee (B \neq \emptyset)$	moving negation inward
(4)	$(A \neq \emptyset) \vee (B \neq \emptyset)$	moving negation inward

We continue the proof using a proof by cases. Expression (4) states that, at least one of the following cases must hold and both can also hold:

(a) There is at least an element in A, assume we have $x \in A$

(b) There is at least an element in B, assume we have $y \in B$

Case (a) above: $x \in A \Rightarrow x \in \{A \cup B\}$ by definition of set union $A \cup B \neq \emptyset$, which contradicts the premise (1).

Case (b) can be shown to yield the same contradiction (exchanging A for B in the above case).

WLOG, we can conclude that

$$A \cup B = \emptyset \Rightarrow (A = \emptyset) \wedge (B = \emptyset)$$

Proving the implication in the opposite direction is straightforward by definition of set union:

$$(A = \emptyset) \wedge (B = \emptyset) \Rightarrow A \cup B = \emptyset$$

□

- Now let's look at functions, say we have the following function: $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \lfloor \frac{x}{2} \rfloor$
  - First what does the graph of this function look like?
  - is f one-to-one (i.e., injective)? No, for example both 1 and 1.1 are assigned 0.
  - Is f onto $\mathbb{R}$ (i.e., surjective)? No, the floor function only maps to integers, so only integers would be mapped to.
- Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $C = \{2, 7, 10\}$

  Consider the following two functions: $g : A \to B$ and $f : B \to C$ where $g : \{(1, b), (2, a), (3, a), (4, b)\}$ and $f : \{(a, 10), (b, 7), (c, 2)\}$
  - Find $f \circ g$. Answer: $\{(1, 7), (2, 10), (3, 10), (4, 7)\}$
  - Findf^{-1}. Answer: $\{(10, a), (7, b), (2, c)\}$
  - Is g^{-1} a function? Answer: No, because a has two pre-images but in a function, each element of the domain must be mapped to *exactly one* element in the co-domain.

- Find $f \circ f^{-1}$. Answer: $\{(10, 10), (7, 7), (2, 2)\}$

- Prove or disprove: $\forall x, y \in \mathbb{R}, \lfloor x \times y \rfloor \leq \lfloor x \rfloor \times \lfloor y \rfloor$
  - let $x = 3.5$, and $y = 1.5$. $\lfloor 3.5 \times 1.5 \rfloor = \lfloor 5.25 \rfloor = 5$, but $\lfloor 3.5 \rfloor \times \lfloor 1.5 \rfloor = 4$
  - $5 \neq 4$, therefore the statement does not hold. This is a proof with a counterexample.

- Prove or disprove for all $x, y \in \mathbb{R}, \lceil x \times y \rceil \leq \lceil x \rceil \times \lceil y \rceil$
  - Here the same example works $\lceil 3.5 \times 1.5 \rceil = \lceil 5.25 \rceil = 6$, but $\lceil 3.5 \rceil \times \lceil 1.5 \rceil = 4 \times 2 = 8$

- Show that the function $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = |x|$ is not invertible, but if the domain is restricted to the set of nonnegative real numbers (i.e., $f : \mathbb{R}^+ \to \mathbb{R}^+$), the resulting function is invertible.

  For a function to be invertible, it must be bijective (i.e., a one-to-one correspondance).

  Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

  1. Injective: No, $f(x_1) \neq f(x_2) \Rightarrow |x_1| \neq |x_2| \Rightarrow \pm x_1 \neq \pm x_2$
     Now, if the domain is restricted to the set of nonnegative real numbers. Is $f(x)$ injective?
     $f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2$. Therefore, on the restricted domain $f(x)$ is injective.
  2. Surjective: Every element in codomain ($\mathbb{R}^+$) is a positive number, then for $\forall b \in codomain(f) \exists a \in \mathbb{R} b = |a|$. Thus, b has necessarily a preimage. Thus, the range and the codomain are equal, we can conclude that f is surjective.
  3. Bijective: No, because it is not injective.
     However, on the restricted domain, it is bijective because it is both injective and surjective.
  4. Invertible: Again, only on the restricted domain.

- Now a quick review of membership, determine whether these statements are true or false:
  1. $\{a, b\} \subseteq \{\{a, b\}\}$
     False, because neither a nor b is an element in $\{\{a, b\}\}$.
  2. $\{a, b\} \in \{\{a, b\}\}$
     True, because there the element $\{a, b\}$ is in $\{\{a, b\}\}$
  3. $\{a, b, c\} \subset \{a, b, c\}$
     False, because the sets are equal, and the statement is wondering if it is a strict subset.

4. $\{a, b, c\} \subseteq \{a, b, c\}$

   True, because the sets are equal.

5. $\{\} \subseteq \{a, b, c\}$

   True, because the empty set $\emptyset = \{\}$ is a subset of all sets.

6. $\emptyset \in \{a, b, c\}$

   False, because the element $\emptyset$ is not in the set $\{a, b, c\}$.

7. $\{a\} \subset \{a, a\}$ *Trick question: watch out!*

   False, the set $\{a, a\}$ is really $\{a\}$ because, in a set, elements are *not* repeated. Therefore, $\{a\} \subset \{a, a\}$ is *false* because $\{a\} \not\subset \{a\}$ (Note that $\{a\} \subseteq \{a\}$ though).