Introduction to NP-Complete Problems

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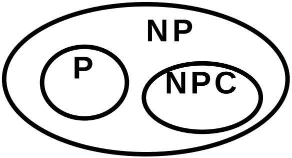
$\mathcal{P}$, $\mathcal{NP}$ and $\mathcal{NP}$-Complete

Given a problem, it belongs to $\mathcal{P}$, $\mathcal{NP}$ or $\mathcal{NP}$-Complete classes, if:

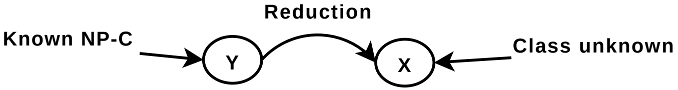
- $\mathcal{NP}$: verifiable in polynomial time.
- $\mathcal{P}$: decidable in polynomial time.
- $\mathcal{NP}$-Complete: all problems in $\mathcal{NP}$ can be reduced to it in polynomial time.



The 4-Step Proof

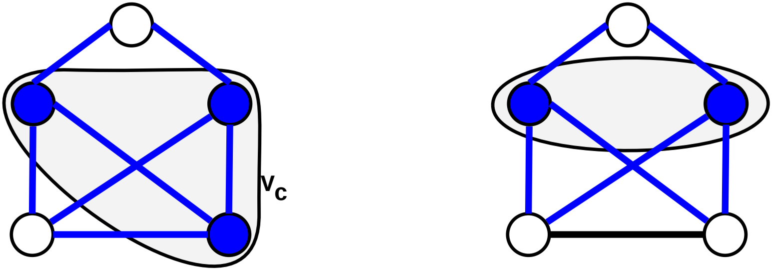
Given a problem X, prove it is in $\mathcal{NP}$-Complete.

1. Prove X is in $\mathcal{NP}$.
2. Select problem Y that is know to be in $\mathcal{NP}$-Complete.
3. Define a polynomial time reduction from Y to X.
4. Prove that given an instance of Y, Y has a solution *iff* X has a solution.



Vertex Cover

- A *vertex cover* of a graph $G = (V, E)$ is a $V_C \subseteq V$ such that every $(a, b) \in E$ is incident to at least a $u \in V_C$.



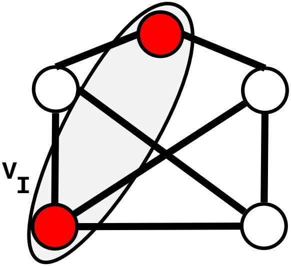
$\longrightarrow$ Vertices in V_C 'cover' all the edges of G.

- The VERTEX COVER (VC) decision problem:

  Does G have a vertex cover of size k?

Independent Set

- An *independent set* of a graph $G = (V, E)$ is a $V_I \subseteq V$ such that no two vertices in V_I share an edge.



$\longrightarrow$ u, $v \in V_I$ cannot be neighbors.

- The INDEPENDENT SET (IS) decision problem:
  Does G have an independent set of size k?

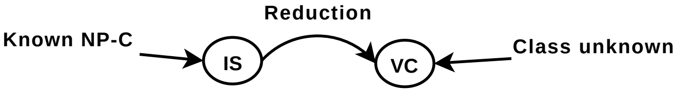
Prove VERTEX COVER is $\mathcal{NP}$-complete

Given that the INDEPENDENT SET (IS) decision problem is $\mathcal{NP}$-complete, prove that VERTEX COVER (VC) is $\mathcal{NP}$-complete.
Solution:

1. Prove VERTEX COVER is in $\mathcal{NP}$.
   - Given V_C, vertex cover of $G = (V, E)$, $|V_C| = k$
   - We can check in $O(|E| + |V|)$ that V_C is a vertex cover for G. How?
     - For each vertex $\in V_C$, remove all incident edges.
     - Check if all edges were removed from G.
   - Thus, VERTEX COVER $\in \mathcal{NP}$

Prove Vertex Cover is $\mathcal{NP}$-complete (2)

2. Select a known $\mathcal{NP}$-complete problem.
   - Independent Set (IS) is a known $\mathcal{NP}$-complete problem.
   - Use IS to prove that VC is $\mathcal{NP}$-complete.



Prove VERTEX COVER is $\mathcal{NP}$-complete (3)

3. Define a polynomial-time reduction from IS to VC:
   - Given a general instance of IS: $G'=(V', E')$, k'
   - Construct a specific instance of VC: $G=(V, E)$, k
     - $V=V'$
     - $E=E'$
     - $(G=G')$
     - $k=|V'| - k'$
   - This transformation is polynomial:
     - Constant time to construct $G=(V, E)$
     - $O(|V|)$ time to count the number of vertices
   - Prove that there is a V_I ($|V_I| = k'$) for G' iff there is an V_C ($|V_C| = k$) for G.

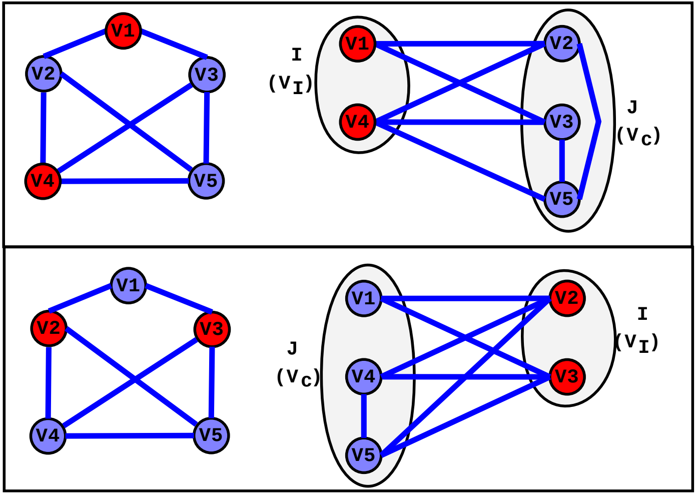
Prove VERTEX COVER is $\mathcal{NP}$-complete (4)

Prove G' has an independent set V_I of size k' *iff* VC has a vertex cover V_C of size k.

- Consider two sets I and J s.t. $I \cap J = \emptyset$ and $I \cup J = V = V'$
- Given any edge (u, v), one of the following four cases holds:
  1. $u, v \in I$
  2. $u \in I$ and $v \in J$
  3. $u \in J$ and $v \in I$
  4. $u, v \in J$
- Assume that I is an independent set of G' then:
  - Case 1 cannot be; (vertices in I cannot be adjacent)
  - In cases 2 and 3, (u, v) has *exactly one* endpoint in J.
  - In case 4, (u, v) has *both* endpoints in J.
  - In cases 2, 3 and 4, (u, v) has *at least one* endpoint J.
  - Thus, vertices in J cover all edges of G'.
  - Also: $|I| = |V| - |J|$ since $I \cap J = \emptyset$ and $I \cup J = V = V'$
  - Thus, if I is an independent set of G', then J is a vertex cover of $G'(= G)$.

Similarly, we can prove that if J is a vertex cover for G', then I is an independent set for G'. □

Prove VERTEX COVER is $\mathcal{NP}$-complete (5)



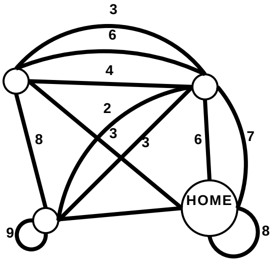
Jogger Problem

Given a weighted, undirected graph G with:

- loops, multiple edges, and only positive weights,
- a special node v called *home*,
- and given an integer $i \geq 0$.

Is there a route for a jogger J that:

- starts from home,
- travels a distance i, and
- returns home
- without repeating an edge (nodes can be repeated)?



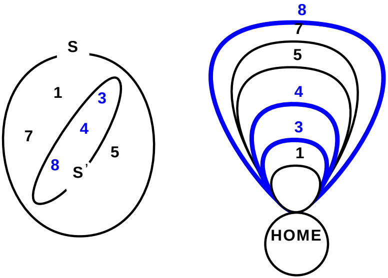
Jogger is $\mathcal{NP}$-complete

1. Jogger is in $\mathcal{NP}$: Given a path P, we can check in $O(|P|)$ whether or not the sum of all edge weights is equal to i.
2. Consider the Subset Sum (SS) problem[1], which is a known $\mathcal{NP}$-complete problem.
   Given a set S of positive integers, is there a subset $S' \subseteq S$ such that sum of the elements of S' is t.
   Example: $S = \{1, 3, 4, 5, 7, 8\}$, find $S' \subseteq S$ such that sum of the elements in S' is 15.
3. Reduce Subset Sum to the Jogger.

[1] One type of knapsack problem.

Jogger is $\mathcal{NP}$-complete

- Given an instance of SS: $S = \{a_1, a_2, \ldots a_n\}$, construct a graph G as follows:
  - G has a unique node, v, which is the home.
  - For each $a_i \in S$, add a self-loop to v of weight a_i.
  - Let i (of the Jogger) $= t$ (of the Sum Set).
- This construction is obviously linear in the number of elements in S.



Jogger is $\mathcal{NP}$-complete (2)

4. SS has a solution *iff* Jogger has a solution.
   - G contains a path starting from home, never repeating an edge, and returning back home with a total distance exactly i *iff* S has a subset S' with sum of elements of S' equal to t.
   - If $S' \subseteq S$ is a solution to SS, then the Jogger has a path of length $i = t$ by taking the edges (loops) corresponding to the elements in S'.
   - If there a path P is a solution to Jogger, then the subset of S with elements corresponding to the edges in P is a subset with sum $i = t$ and thus is a solution to SS. $\square$