Recitation 12

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- 7.1.9d) Use backwards substitution to solve $a_n = a_{n-1} + 2n + 3$, when $a_0 = 4$.

$$\begin{aligned}
a_n &= a_{n-1} + 2n + 3 \\
&= (a_{n-2} + 2(n-1) + 3) + 2n + 3 \\
&= a_{n-2} + 2n - 2 + 3 + 2n + 3 \\
&= a_{n-2} + 4n + 2(3) - 2 \\
&= (a_{n-3} + 2(n-2) + 3) + 4n + 2(3) - 2 \\
&= a_{n-3} + 2n - 4 + 3 + 4n + 2(3) - 2 \\
&= a_{n-3} + 6n + 3(3) - 2 - 4 \\
&= (a_{n-4} + 2(n-3) + 3) + 6n + 3(3) - 2 - 4 \\
&= a_{n-4} + 2n - 6 + 3 + 6n + 3(3) - 2 - 4 \\
&= a_{n-4} + 8n + 4(3) - 2 - 4 - 6 \\
&= \ldots \\
&= a_{n-n} + 2n(n) + n(3) - 2 - 4 - 6 - \ldots - 2(n-1) \\
&= a_0 + 2n^2 + 3n - \sum_{i=1}^{n-1} 2i \\
&= a_0 + 2n^2 + 3n - 2\frac{(n-1)(n-1+1)}{2} \\
&= a_0 + 2n^2 + 3n - n(n-1) \\
&= a_0 + 2n^2 + 3n - n^2 + n \\
&= n^2 + 4n + 4
\end{aligned}$$

- Find the solution to the recurrence relation:

$$\begin{aligned}
a_n &= 2a_{n-1} + 8a_{n-2} \\
a_0 &= 3 \\
a_1 &= 4
\end{aligned}$$

This relation is a linear homogeneous recurrence relation of degree 2 because:

- The right-hand side has only multiples of previous terms of the sequence and coefficients are all constants. Therefore, it is linear.
- No terms occur that are not multiples of a_j (i.e., no non-recursive cost). Therefore, it is homogeneous.
- It is expressed in terms of the $(n-2)^{th}$ term of the sequence. Therefore, it is of degree 2).

We know that a solution to solve this recurrence relation is of the form $a_n = r^n$ where r is some real constant. Replacing the solution in the recurrence relation, we get:

$$r^n = 2r^{n-1} + 8r^{n-2}.$$

Dividing by r^{n-2}, we get:

$$r^2 = 2r^1 + 8.$$

Thus, the *characteristic equation* of this recurrence relation is:

$$r^2 - 2r - 8 = (r+2)(r-4) = 0.$$

This characteristic equation has the roots $r_1 = -2$ and $r_2 = 4$; Therefore, the solution of the recurrence relation is

$$a_n = \alpha_1(-2)^n + \alpha_2 4^n.$$

Plugging in our initial conditions we get

$$\begin{aligned} 3 &= \alpha_1 + \alpha_2 \\ 4 &= -2\alpha_1 + 4\alpha_2 \end{aligned}$$

Solving for $\alpha_1 = 3 - \alpha_2$, we get $4 = -2(3-\alpha_2) + 4\alpha_2 \Rightarrow 4 = -6 + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{5}{3} = \alpha_2$. Therefore, $\alpha_1 = \frac{4}{3}$ and $\alpha_2 = \frac{5}{3}$.

Putting the values of α_1, α_2 back in the solution form, we obtain the following solution of the recurrence relation given the boundary conditions

$$a_n = \frac{4}{3}(-2)^n + \frac{5}{3}4^n.$$

- *You are not responsible for solving non-homogeneous recurrence relations.*
  Solve the following linear non-homogeneous recurrence relation:

$$\begin{aligned} a_n &= 2a_{n-1} - 8a_{n-2} + n && (1) \\ a_0 &= 3 && (2) \\ a_1 &= 4 && (3) \end{aligned}$$

We notice that $f(n)$ is polynomial n. We will solve this problem using Theorem 6 on page 469, which covers this case, the case that $f(n)$ is an exponential in n, and the case where $f(n)$ is a product of a polynomial and an exponential in n.

First, we solve the associated linear *homogeneous* recurrence relation, which happens to be the one above :-). Its solution is:

$$a_n = \alpha_1(-2)^n + \alpha_2 4^n.$$

Next, we find a particular solution for the given non-homogeneous term. Theorem 6 applies to $f(n)$ of the form:

$$f(n) = (b_t n^t + b_{t-1}n^{t-1} + \ldots + b_1 n + b_0)s^n.$$

In our case, $f(n) = n$ and s is 1. Since our s is *not* a root of our characteristic equation (*relief*), there is a particular solution of the form:

$$a^p = (p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n.$$

For us, the particular solution is $a^p = p_1 n + p_0$. Plugging the particular solution in Equation (1), we get:

$$\begin{aligned} a^p = p_1 n + p_0 &= 2(p_1(n-1) + p_0) - 8(p_1(n-2) + p_0) + n \\ &= 2p_1 n - 2p_1 + 2p_0 - 8p_1 n + 16p_1 - 8p_0 + n \end{aligned}$$

Moving all terms to one side of the equation, we get:

$$(7p_1 - 1)n + (-14p_1 + 7p_0) = 0.$$

Given that $n \neq 0$, we must have the following:

$$\begin{aligned} 7p_1 - 1 = 0 \\ -14p_1 + 7p_0 = 0 \end{aligned}$$

Now, $7p_1 - 1 = 0 \Rightarrow 7p_1 = 1 \Rightarrow p_1 = \frac{1}{7}$.
Further, $-14p_1 + 7p_0 = 0 \Rightarrow -14(\frac{1}{7}) + 7p_0 = 0 \Rightarrow -2 + 7p_0 = 0 \Rightarrow 7p_0 = 2 \Rightarrow p_0 = \frac{2}{7}$.
We have thus found the particular solution: have $a^p = \frac{1}{7}n + \frac{2}{7}$. Therefore,

$$\begin{aligned} a_n &= a^h + a^p \\ a_n &= (\alpha_1(-2)^n + \alpha_2 4^n) + (\frac{1}{7}n + \frac{2}{7}). \end{aligned}$$

To determine the values of α_1, α_2, we plug in our initial conditions:

$$\begin{aligned} 3 &= \frac{2}{7} + \alpha_1 + \alpha_2 \\ 4 &= \frac{1}{7} + \frac{2}{7} + -2\alpha_1 + 4\alpha_2 \end{aligned}$$

Or:

$$\begin{aligned}\frac{19}{7} &= \alpha_1 + \alpha_2 \\ \frac{25}{7} &= -2\alpha_1 + 4\alpha_2\end{aligned}$$

Solving for $\alpha_1 = \frac{19}{7} - \alpha_2$, we get $\frac{25}{7} = -2(\frac{19}{7} - \alpha_2) + 4\alpha_2 \Rightarrow \frac{25}{7} = -\frac{38}{7} + 2\alpha_2 + 4\alpha_2 \Rightarrow \frac{63}{7} = 6\alpha_2 \Rightarrow \frac{63}{42} = \alpha_2$ Replacing α_2 by its value, we get $\alpha_1 = \frac{-13}{42}$.
Replacing α_1, α_2, we get

$$a_n = \frac{1}{7}n + \frac{2}{7} + \frac{63}{42}(-2)^n + \frac{-13}{42}4^n.$$

- Give the asymptotic characterization for $T(n) = 3T(n/4) + 8n^3$.

  Remember Master Theorem, when we have $T(n) = aT(n/b) + f(n)$ where:

  - $T(n)$ is monotone
  - $f(n) \in \Theta(n^d)$ where $d \geq 0$,
  - b is a constant

  we can use it to classify our recurrence relation as follows:

  $$T(n) \text{ is } \begin{cases} \Theta(n^d) & a < b^d \\ \Theta(n^d \log n) & a = b^d \\ \Theta(n^{\log_b a}) & a > b^d \end{cases}$$

  Therefore, we can use Master's theorem and $a = 3$, $b = 4$ and $d = 3$. Therefore, $T(n)$ is $\mathcal{O}(n^3)$ because $a < b^d(3 < 64)$.