

Introduction to the Boolean Satisfiability Problem

Spring 2019

CSCE 235H Introduction to Discrete Structures

URL: cse.unl.edu/~cse235h

All questions: [Piazza](#)

Satisfiability Study

- 7 weeks
- 30 min lectures in recitation
- ~2 hours of homework per week
- Goals:
 - Exposure to fundamental research in CS
 - Understand how to model problems
 - Learn to use SAT solver, MiniSAT

Boolean Satisfiability Problem

- Given:
 - A Boolean formula
- Question:
 - Is there an assignment of truth values to the Boolean variables such that the formula holds true?

Boolean Satisfiability Problem

$$a \vee (\neg a \wedge b)$$

$$(a \vee \neg a) \rightarrow (b \wedge \neg b)$$

Boolean Satisfiability Problem

$$a \vee (\neg a \wedge b)$$

SATISFIABLE
a=true, b=true

$$(a \vee \neg a) \rightarrow (b \wedge \neg b)$$

Boolean Satisfiability Problem

$$a \vee (\neg a \wedge b)$$

SATISFIABLE
a=true, b=true

$$(a \vee \neg a) \rightarrow (b \wedge \neg b)$$

UNSATISFIABLE
Left side of implication is a tautology.
Right side of implication is a contradiction.
True cannot imply false.

Applications of SAT

- Scheduling
- Resource allocation
- Hardware/software verification
- Planning
- Cryptography

Conjunctive Normal Form

- Variable, term a, b, p, q, x_1, x_2
- Literal $a, \neg a, q, \neg q, x_1, \neg x_1$
- Clause $(a \vee \neg b \vee c)$
- Formula, sentence $(a \vee \neg b \vee c)$
 $\wedge (b \vee c)$
 $\wedge (\neg a \vee \neg c)$

Converting to CNF

- All Boolean formulas can be converted to CNF
- The \rightarrow , \leftrightarrow , \oplus operators can be rewritten in terms of \neg , \vee , \wedge
- \neg , \vee , \wedge can be rearranged using
 - De Morgan's Laws
 - Distributive Laws
 - Double Negative
- May result in exponential size increase of the formula

Converting to CNF

$$(a \vee \neg a) \rightarrow (b \wedge \neg b) \equiv$$

Converting to CNF

$$(a \vee \neg a) \rightarrow (b \wedge \neg b) \equiv$$

Implication $\neg(a \vee \neg a) \vee (b \wedge \neg b) \equiv$

Converting to CNF

$$(a \vee \neg a) \rightarrow (b \wedge \neg b) \equiv$$

Implication $\neg(a \vee \neg a) \vee (b \wedge \neg b) \equiv$

DeMorgan's $(\neg a \wedge a) \vee (b \wedge \neg b) \equiv$

Converting to CNF

$$(a \vee \neg a) \rightarrow (b \wedge \neg b) \equiv$$

Implication $\neg(a \vee \neg a) \vee (b \wedge \neg b) \equiv$

DeMorgan's $(\neg a \wedge a) \vee (b \wedge \neg b) \equiv$

$$(\neg a \vee b) \wedge (\neg a \vee \neg b) \wedge (a \vee b) \wedge (a \vee \neg b)$$

Distributive

Interpretation of CNF

- Every clause must be satisfied by at least one true literal
- Total possible number of solutions increases as number of variables increases
- Clauses constrain the possible solutions
- Smaller clauses are more constraining

Interpretation of CNF

$$(a \vee \neg b \vee \neg c) \\ \wedge (b \vee c) \\ \wedge (\neg a)$$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Interpretation of CNF

$$\begin{aligned} (a \vee \neg b \vee \neg c) &\equiv \\ \wedge (b \vee c) &\rightarrow \neg(\neg a \wedge b \wedge c) \\ \wedge (\neg a) & \end{aligned}$$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Interpretation of CNF

$$(a \vee \neg b \vee \neg c)$$

$$\wedge (b \vee c) \equiv \neg(\neg b \wedge \neg c)$$

$$\wedge (\neg a)$$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Interpretation of CNF

$$(a \vee \neg b \vee \neg c)$$

$$\wedge (b \vee c)$$

$$\wedge (\neg a)$$

\equiv

$$\neg(a)$$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Interpretation of CNF

$$(a \vee \neg b \vee \neg c) \\ \wedge (b \vee c) \\ \wedge (\neg a)$$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Determining SAT/UNSAT

- All that is required to show satisfiability is to find a valid solution
- Many techniques available:
 - Guessing and checking
 - Systematic search
 - Inference

Systematic Search with Backtracking

- Construct a binary tree of all combinations
- Proceeds in a depth first manner
- Each level corresponds to a variable
- Each branch corresponds to a truth assignment
- Branches of the tree are 'pruned' when the assignment cannot be extended in a satisfiable manner

Systematic Search with Backtracking

$$(a \vee b \vee c)$$

$$\wedge(\neg a \vee \neg b)$$

$$\wedge(\neg b \vee \neg c)$$

$$\wedge(\neg c \vee \neg a)$$

root

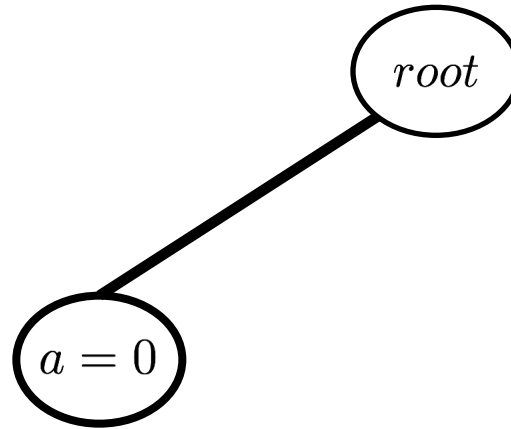
Systematic Search with Backtracking

$$(\underline{a} \vee b \vee c)$$

$$\wedge(\underline{\neg a} \vee \neg b)$$

$$\wedge(\neg b \vee \neg c)$$

$$\wedge(\neg c \vee \underline{\neg a})$$



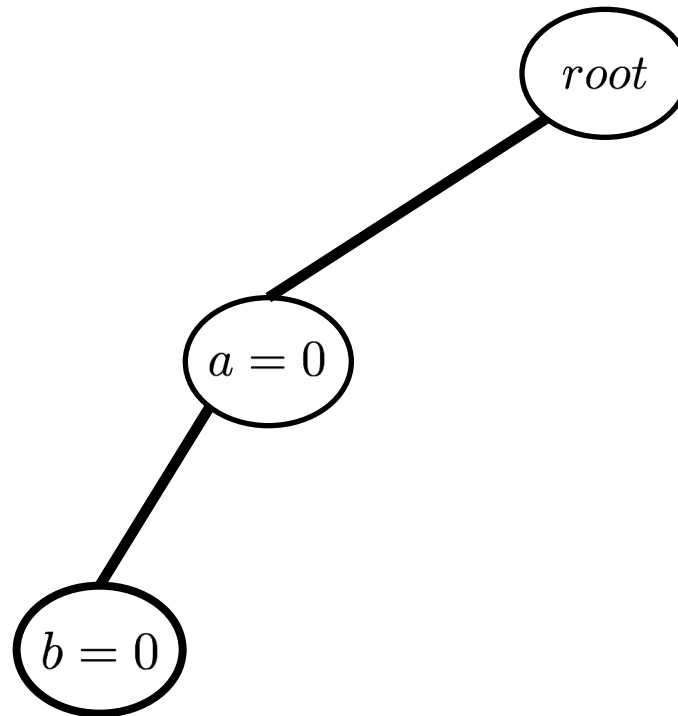
Systematic Search with Backtracking

$$(\underline{a} \vee \underline{b} \vee c)$$

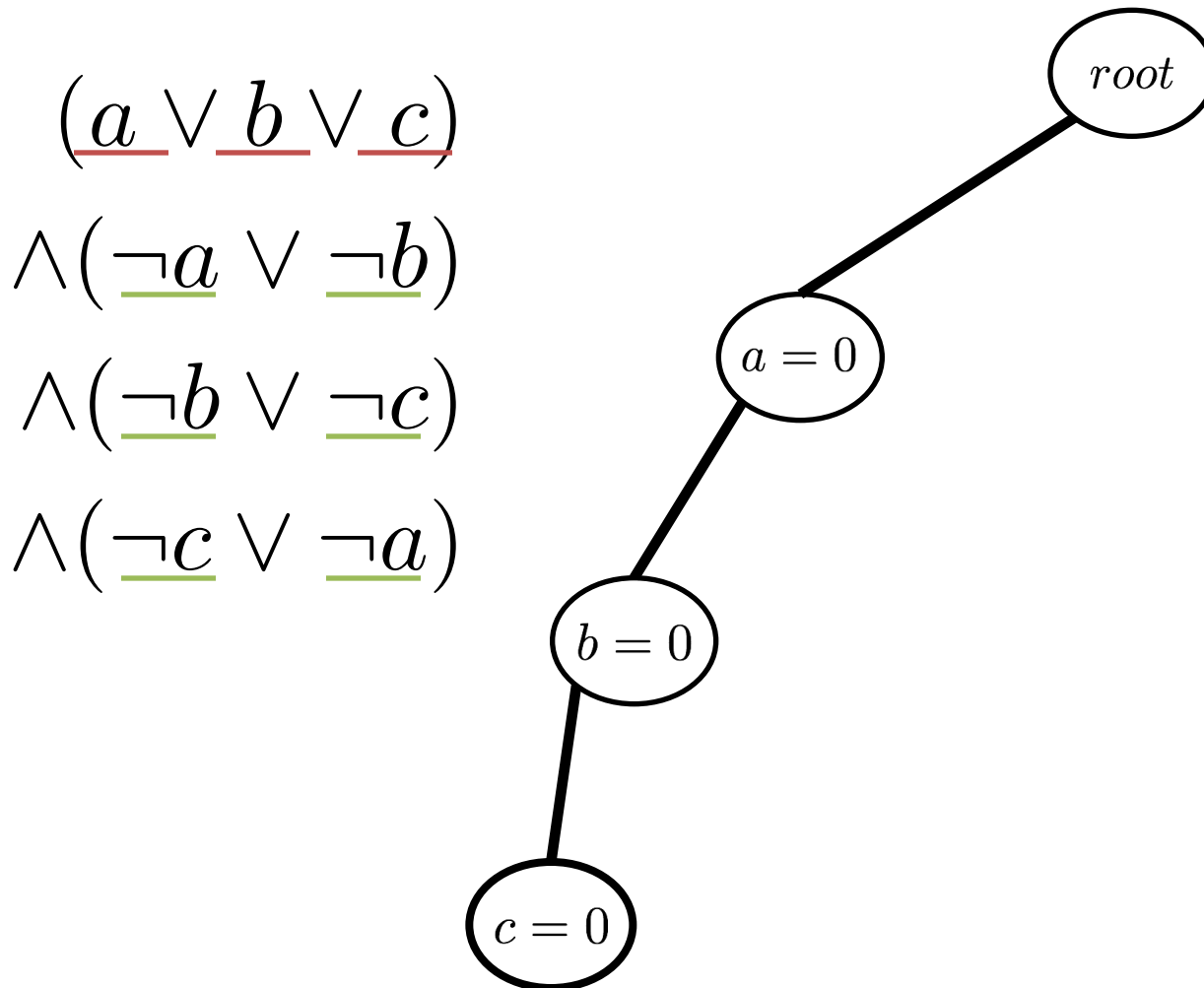
$$\wedge(\underline{\neg a} \vee \underline{\neg b})$$

$$\wedge(\underline{\neg b} \vee \neg c)$$

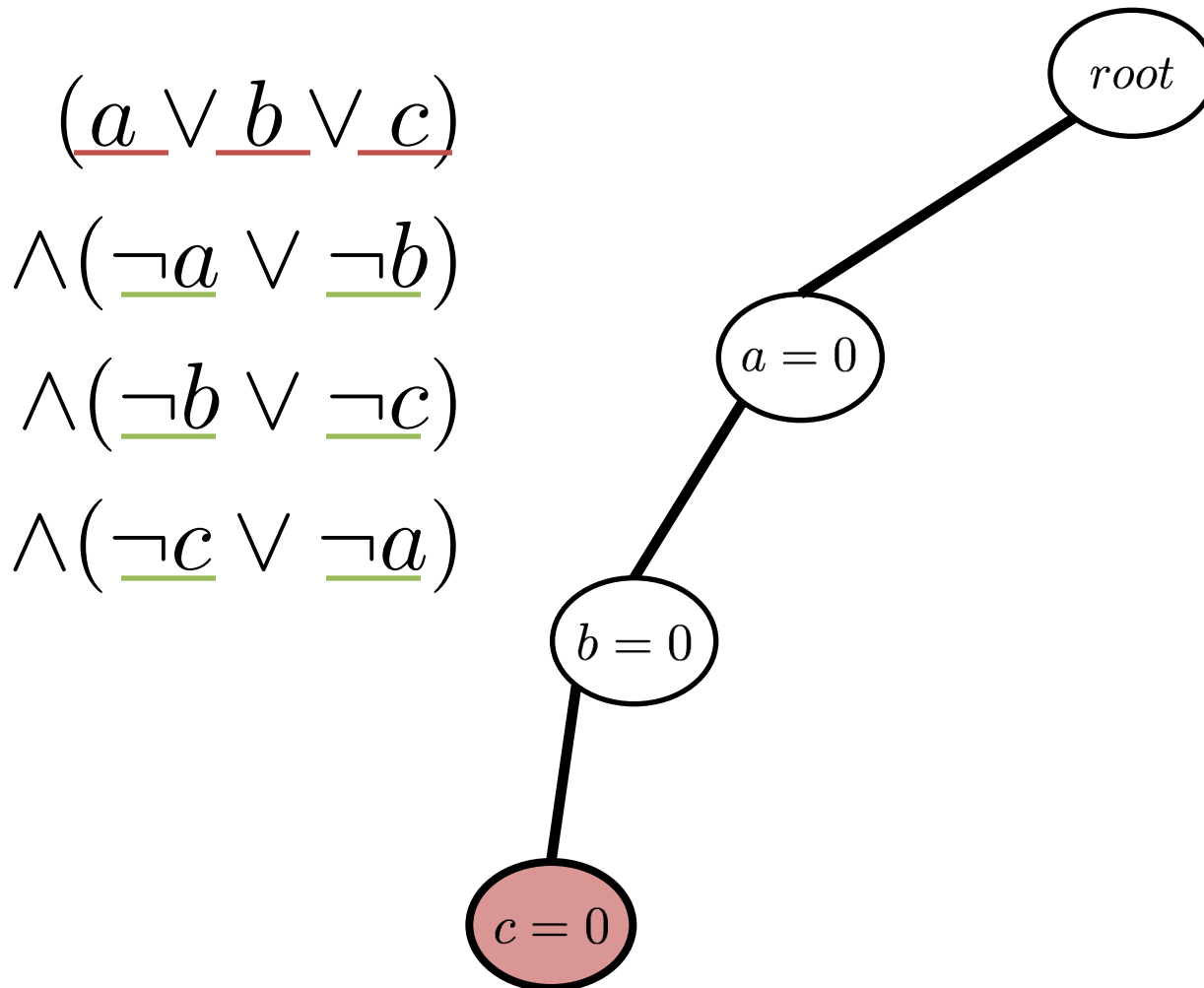
$$\wedge(\neg c \vee \underline{\neg a})$$



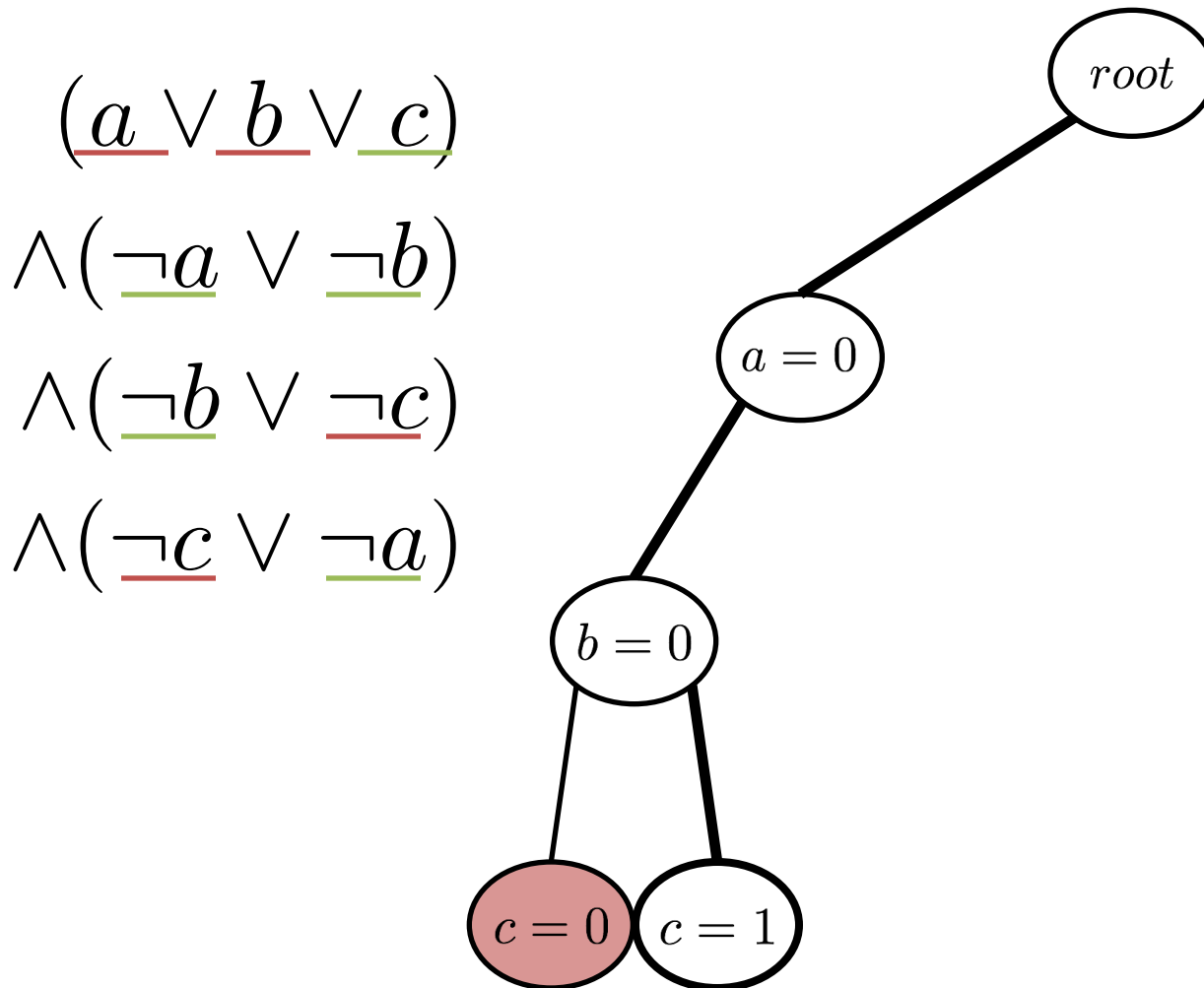
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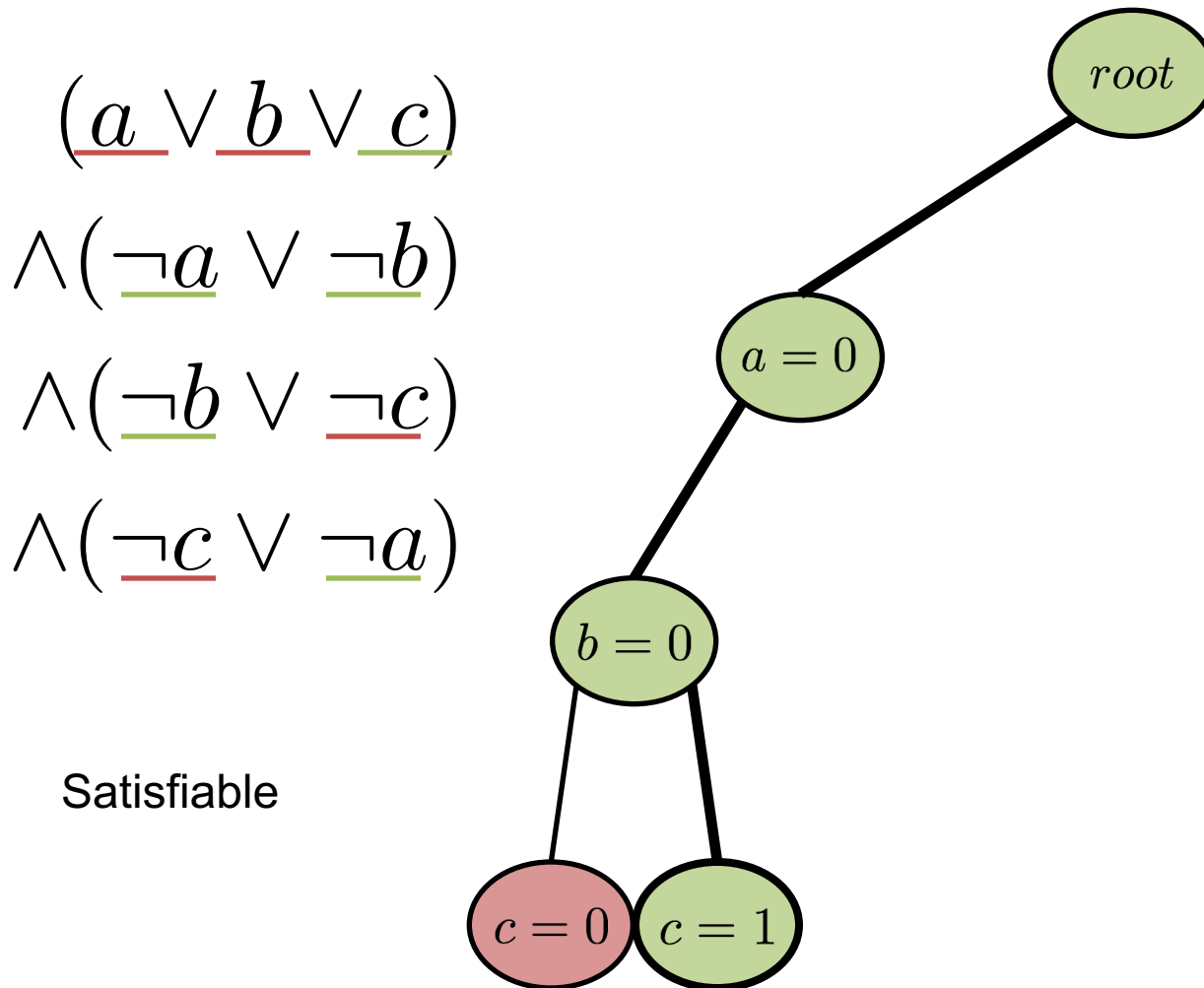
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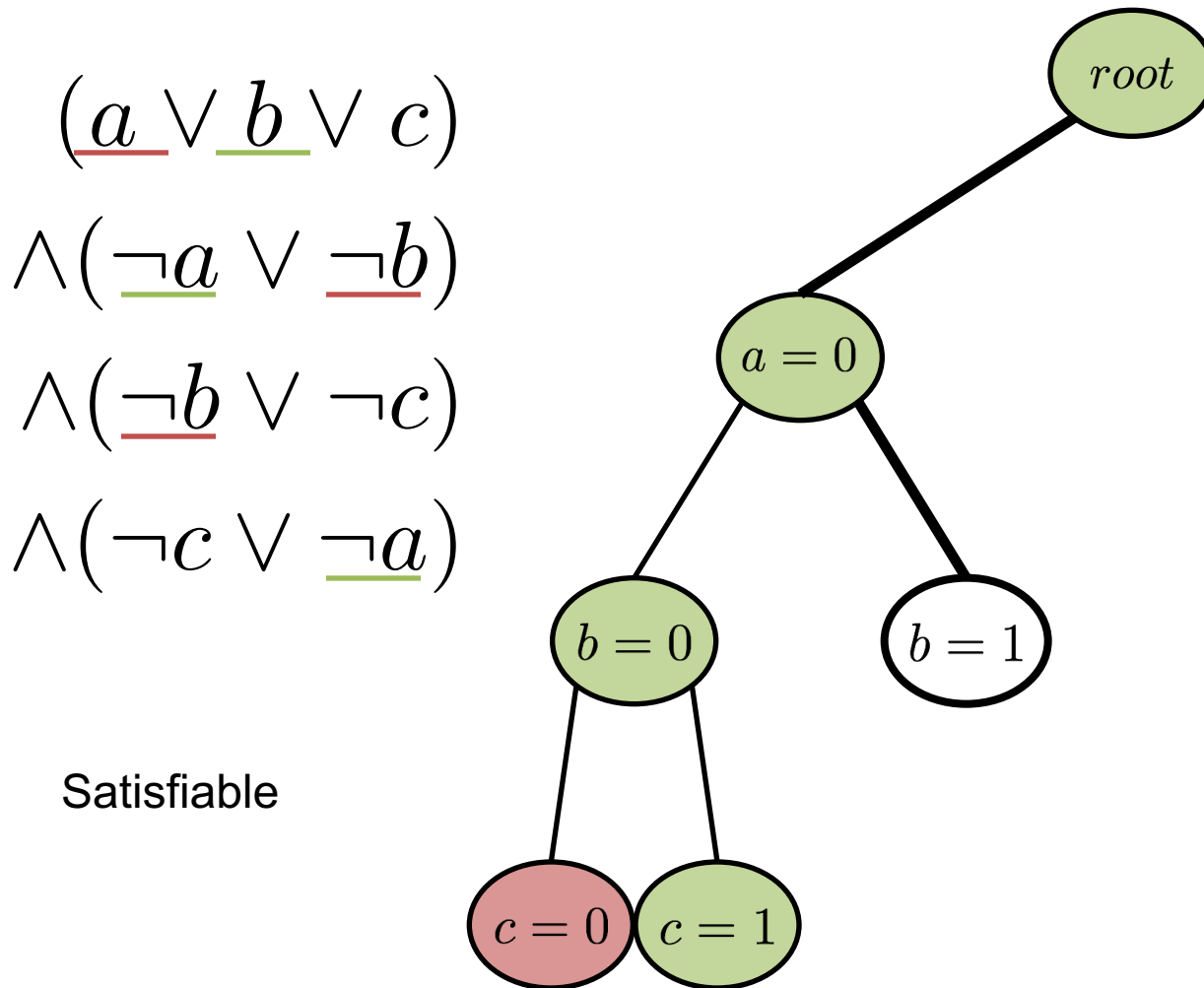
Systematic Search with Backtracking



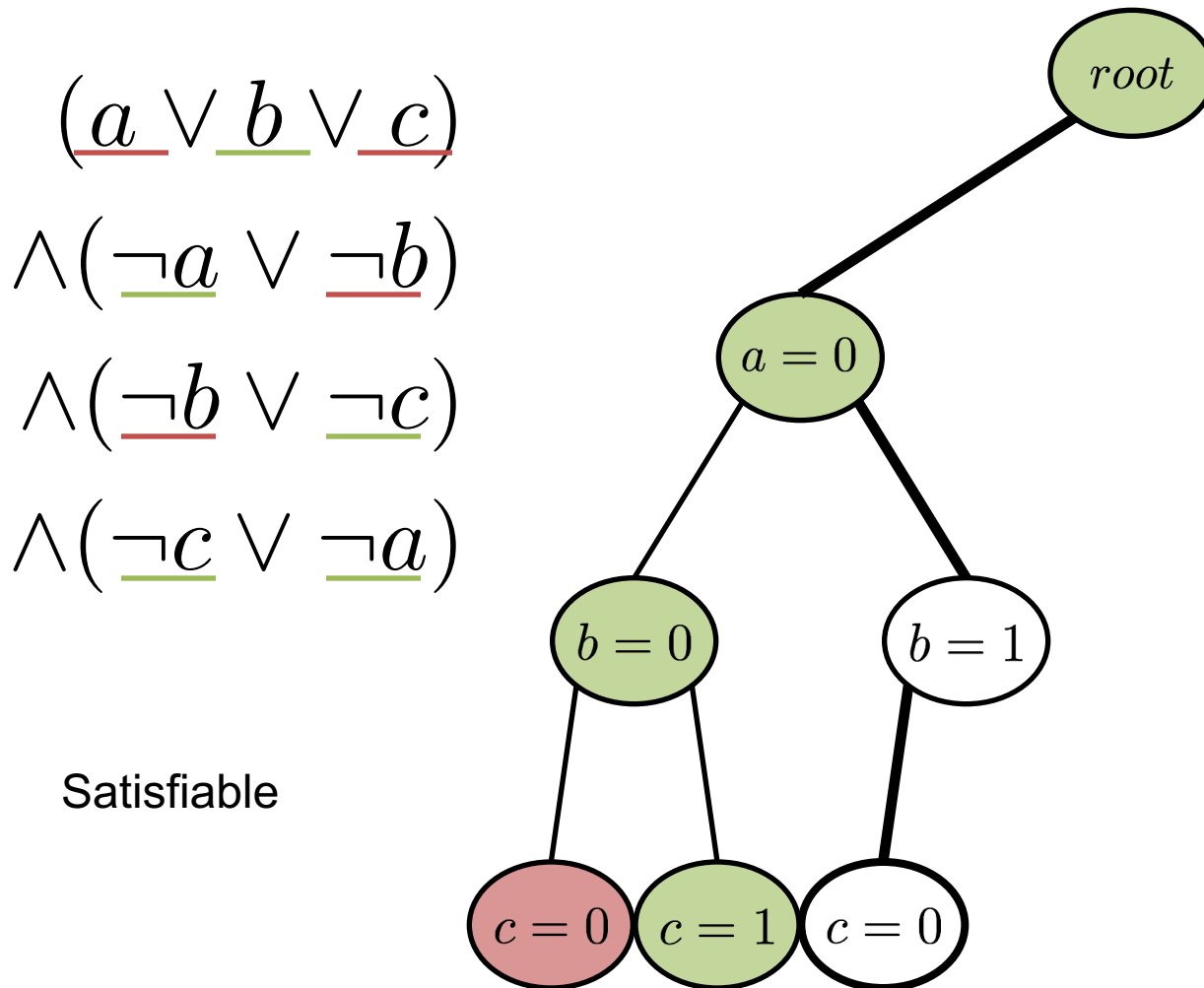
Systematic Search with Backtracking



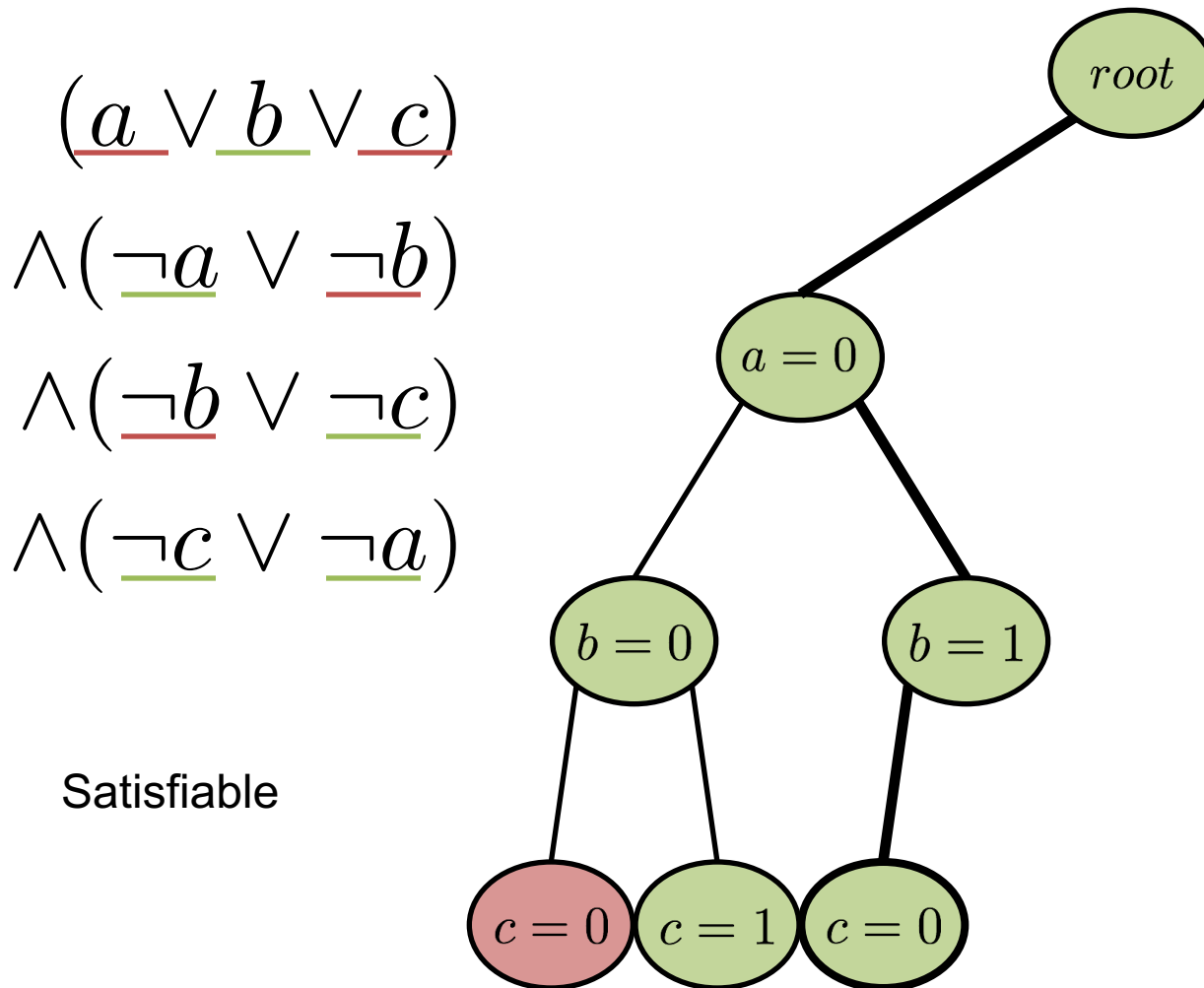
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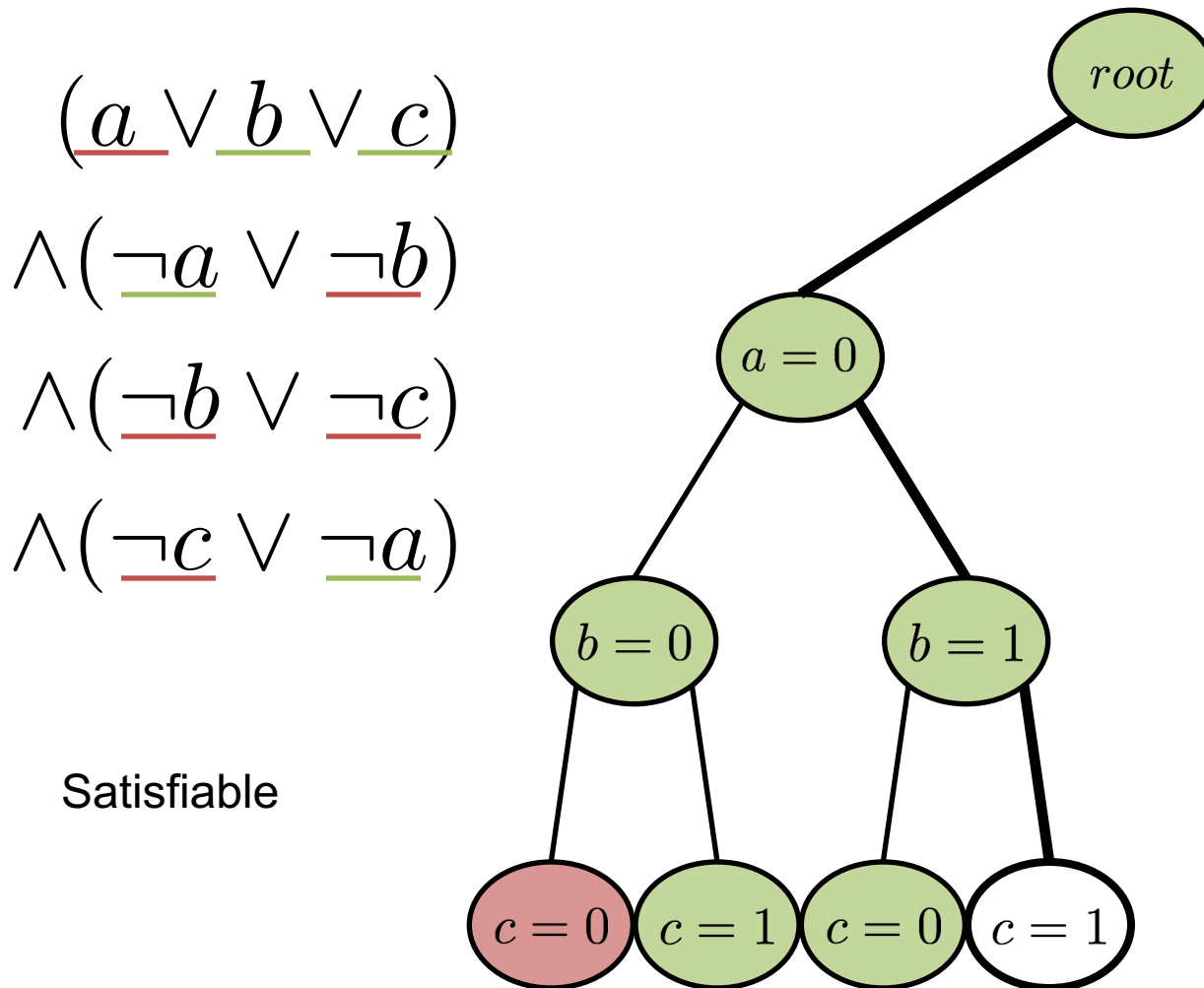
Systematic Search with Backtracking



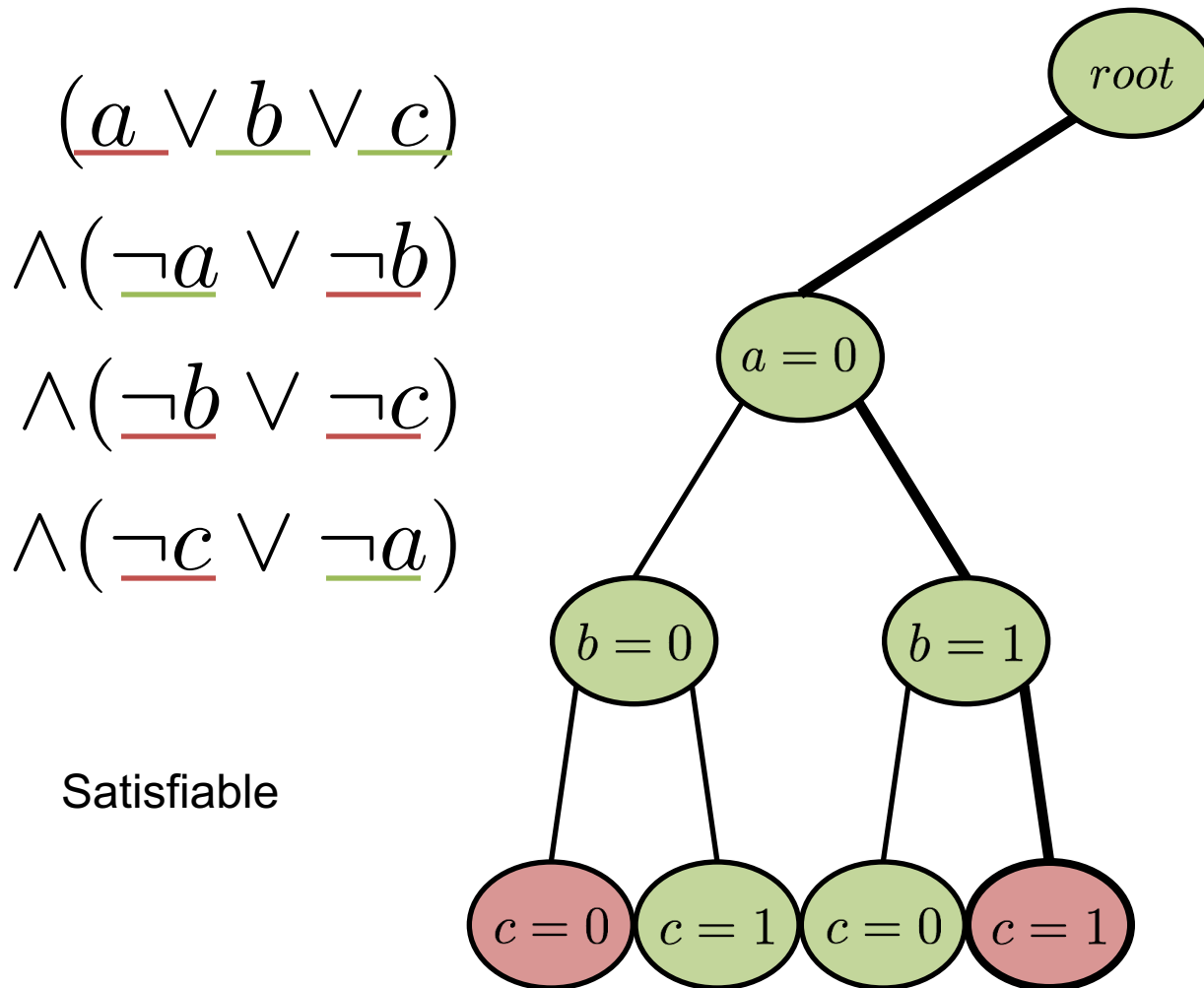
Systematic Search with Backtracking



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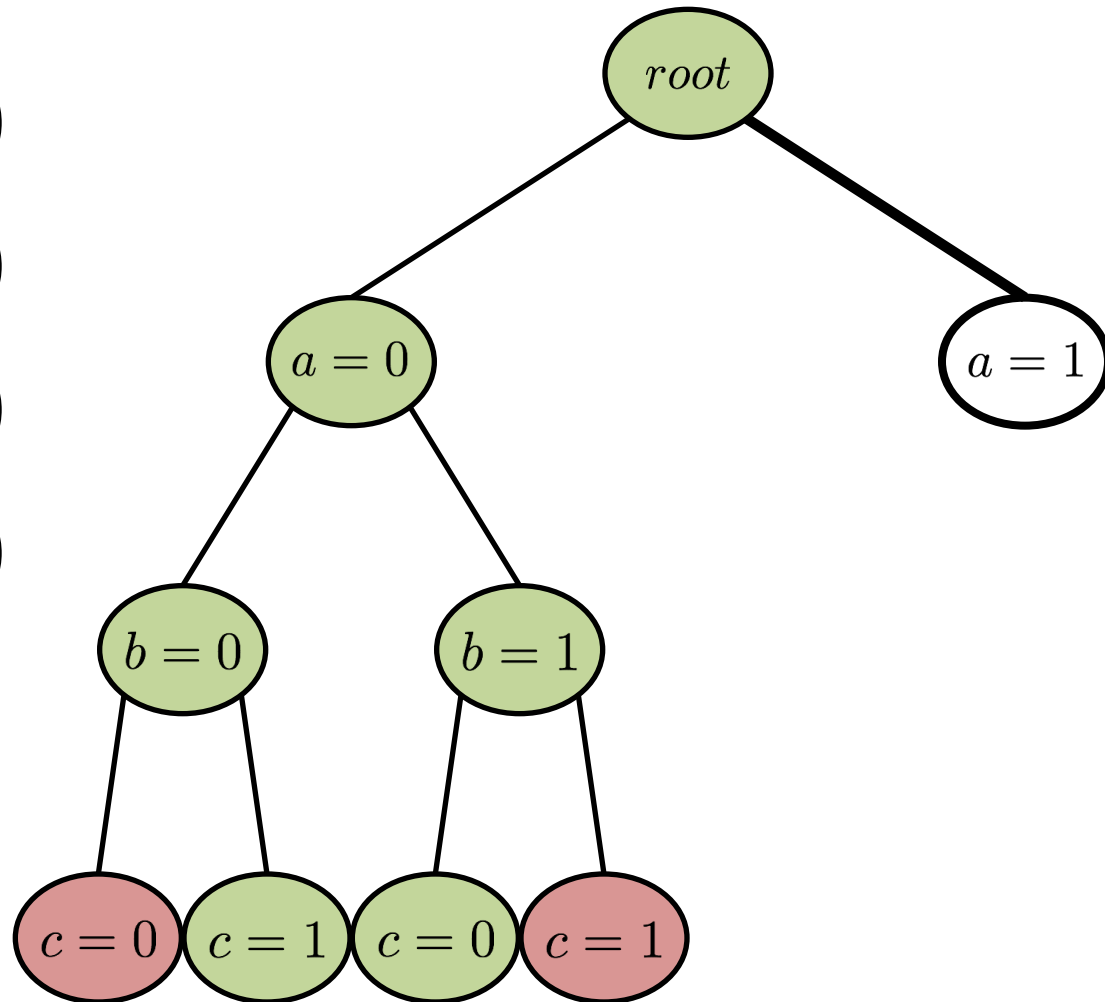
Systematic Search with Backtracking



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee b \vee c) \\ &\wedge(\underline{\neg a} \vee \neg b) \\ &\wedge(\neg b \vee \neg c) \\ &\wedge(\neg c \vee \underline{\neg a}) \end{aligned}$$

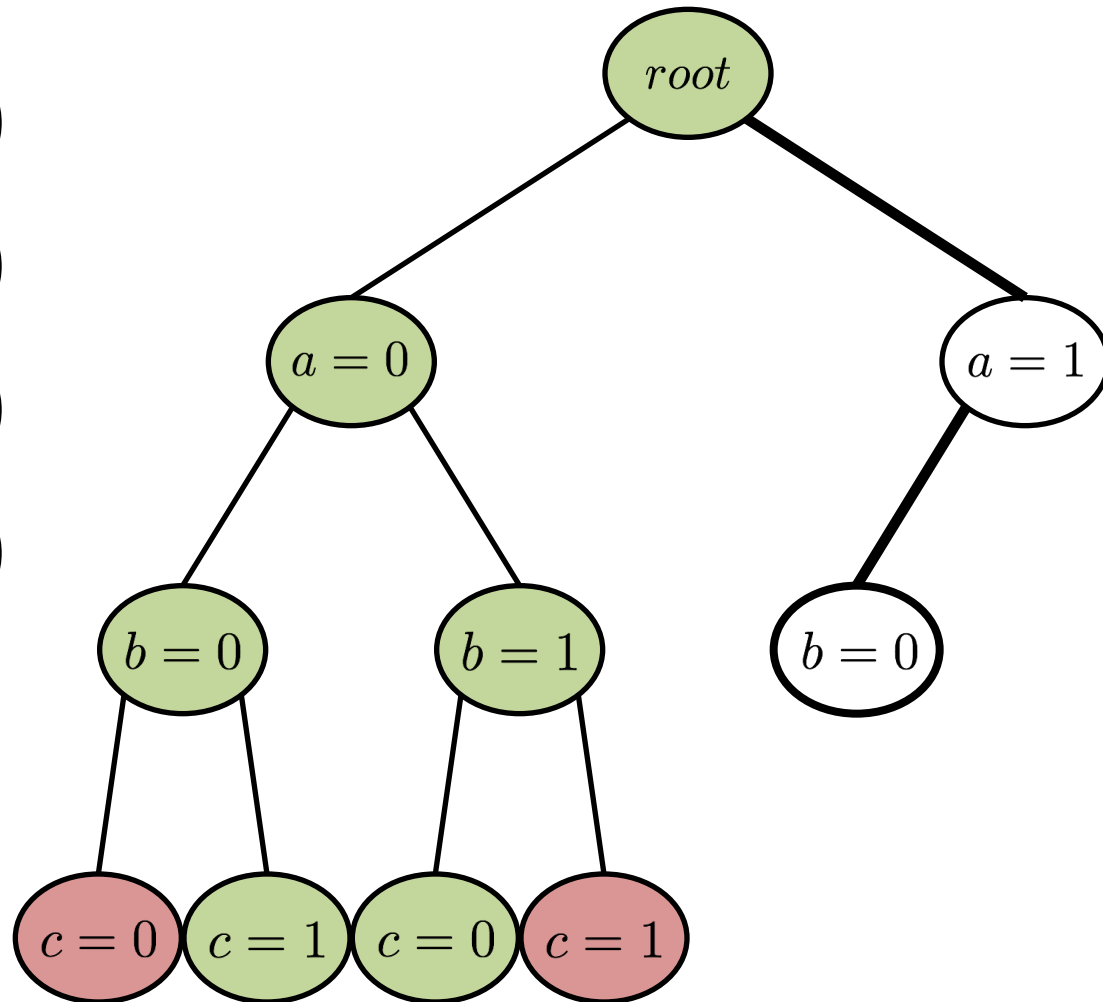
Satisfiable



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee \underline{b} \vee c) \\ &\wedge(\underline{\neg a} \vee \underline{\neg b}) \\ &\wedge(\underline{\neg b} \vee \neg c) \\ &\wedge(\neg c \vee \underline{\neg a}) \end{aligned}$$

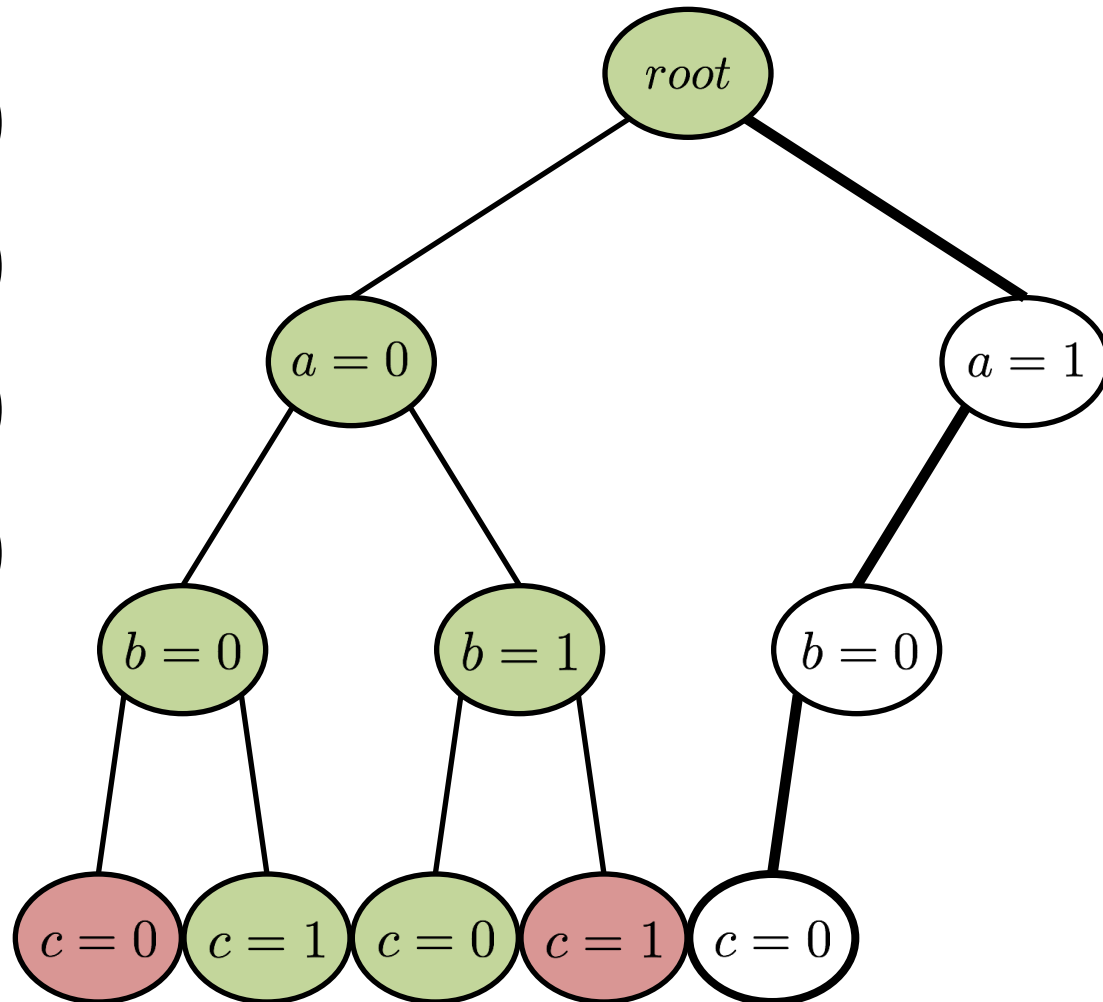
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Systematic Search with Backtracking

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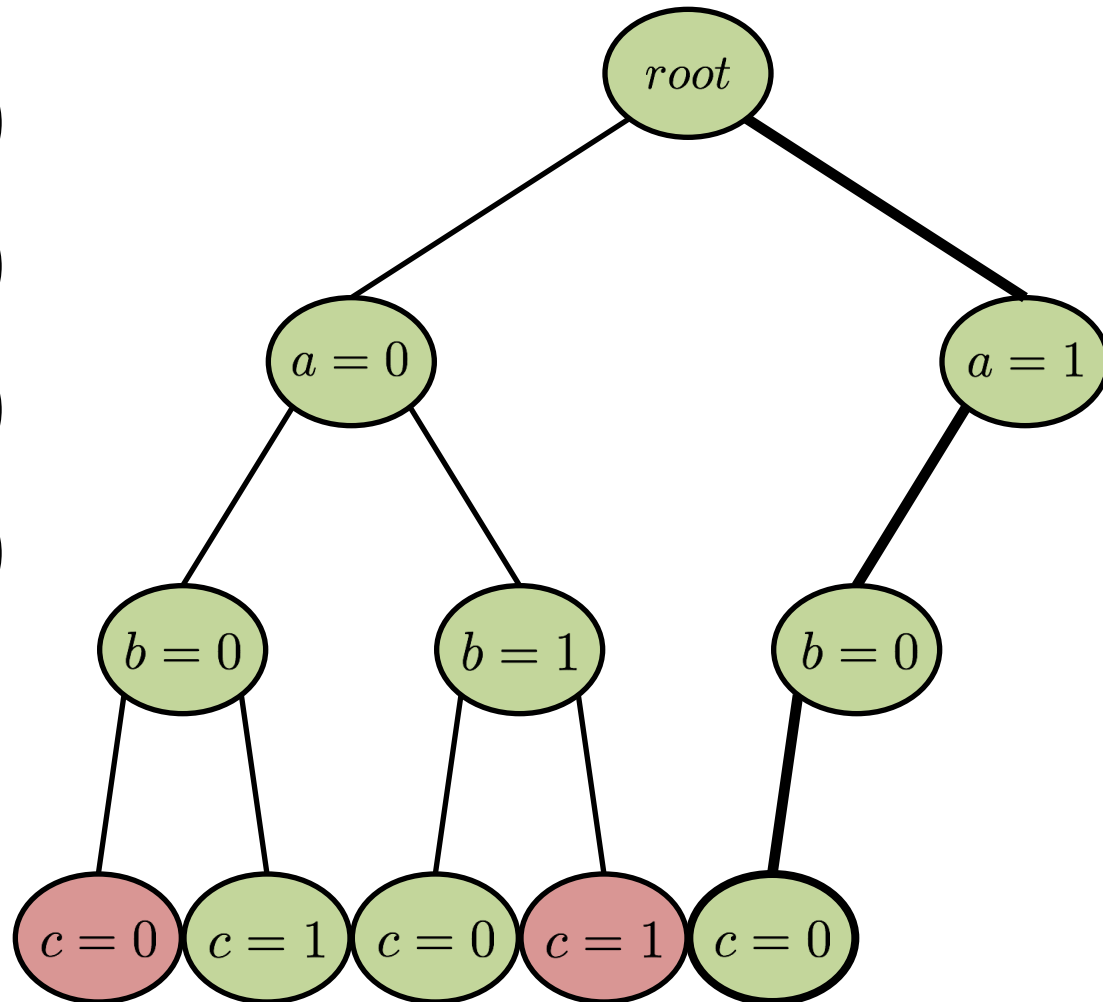
Satisfiable



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee \underline{b} \vee \underline{c}) \\ &\wedge(\underline{\neg a} \vee \underline{\neg b}) \\ &\wedge(\underline{\neg b} \vee \underline{\neg c}) \\ &\wedge(\underline{\neg c} \vee \underline{\neg a}) \end{aligned}$$

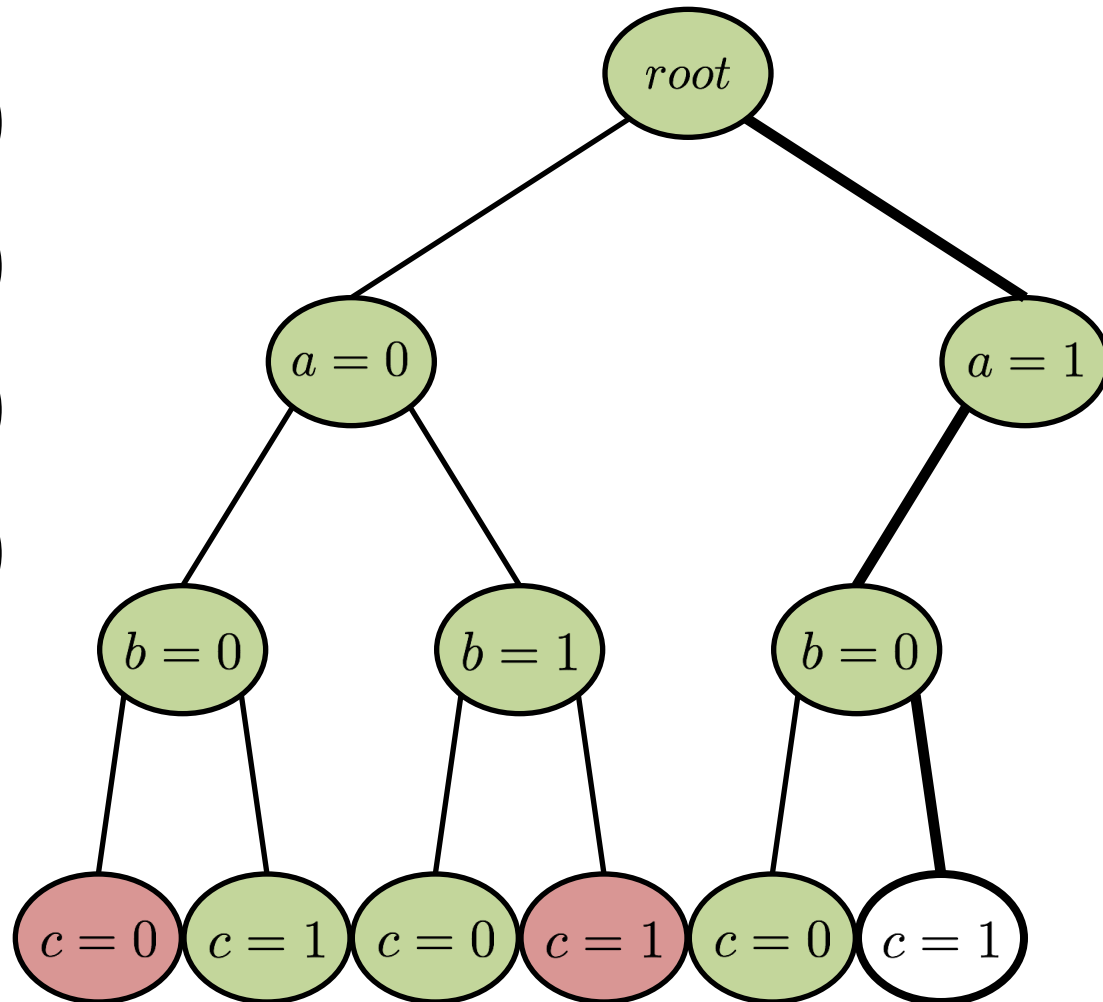
Satisfiable



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee \underline{b} \vee \underline{c}) \\ &\wedge(\underline{\neg a} \vee \underline{\neg b}) \\ &\wedge(\underline{\neg b} \vee \underline{\neg c}) \\ &\wedge(\underline{\neg c} \vee \underline{\neg a}) \end{aligned}$$

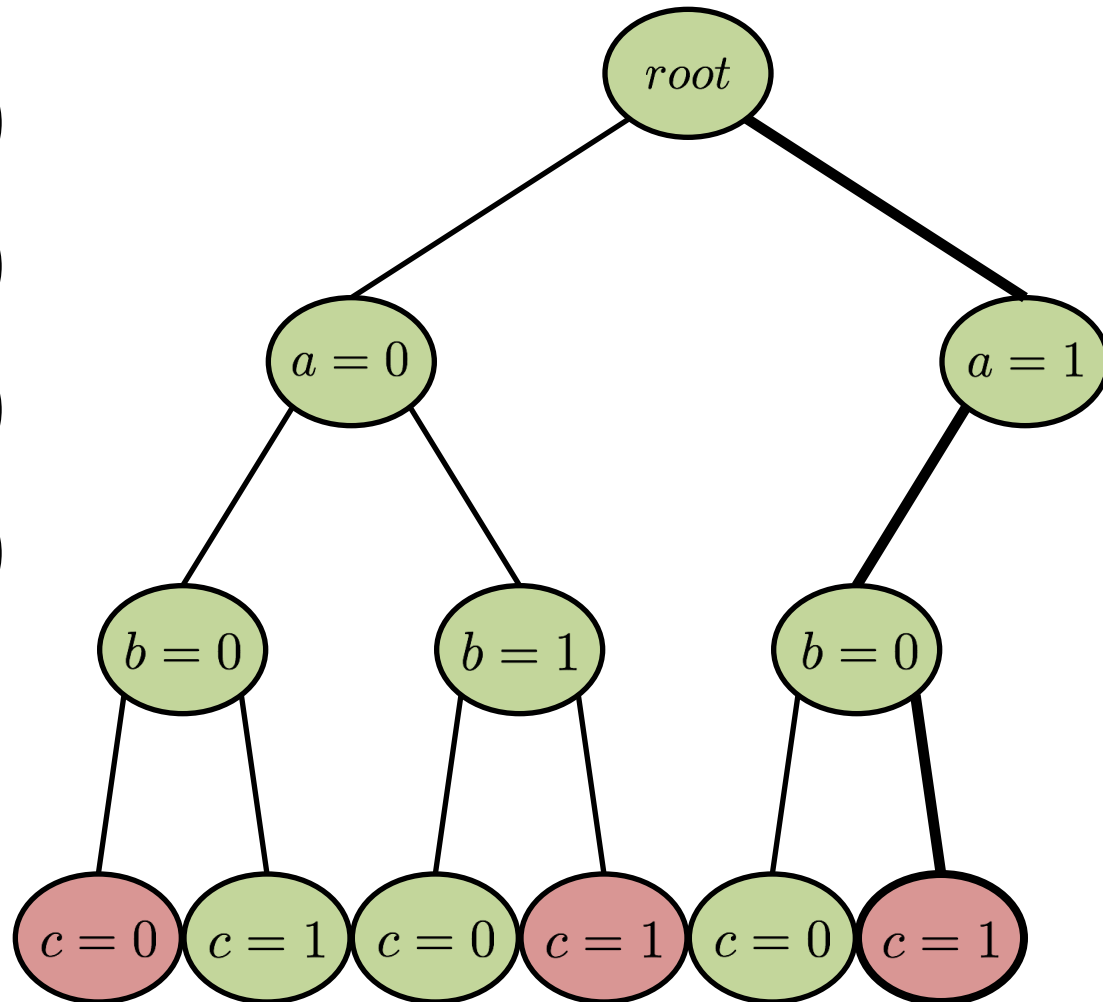
Satisfiable



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee \underline{b} \vee \underline{c}) \\ &\wedge(\underline{\neg a} \vee \underline{\neg b}) \\ &\wedge(\underline{\neg b} \vee \underline{\neg c}) \\ &\wedge(\underline{\neg c} \vee \underline{\neg a}) \end{aligned}$$

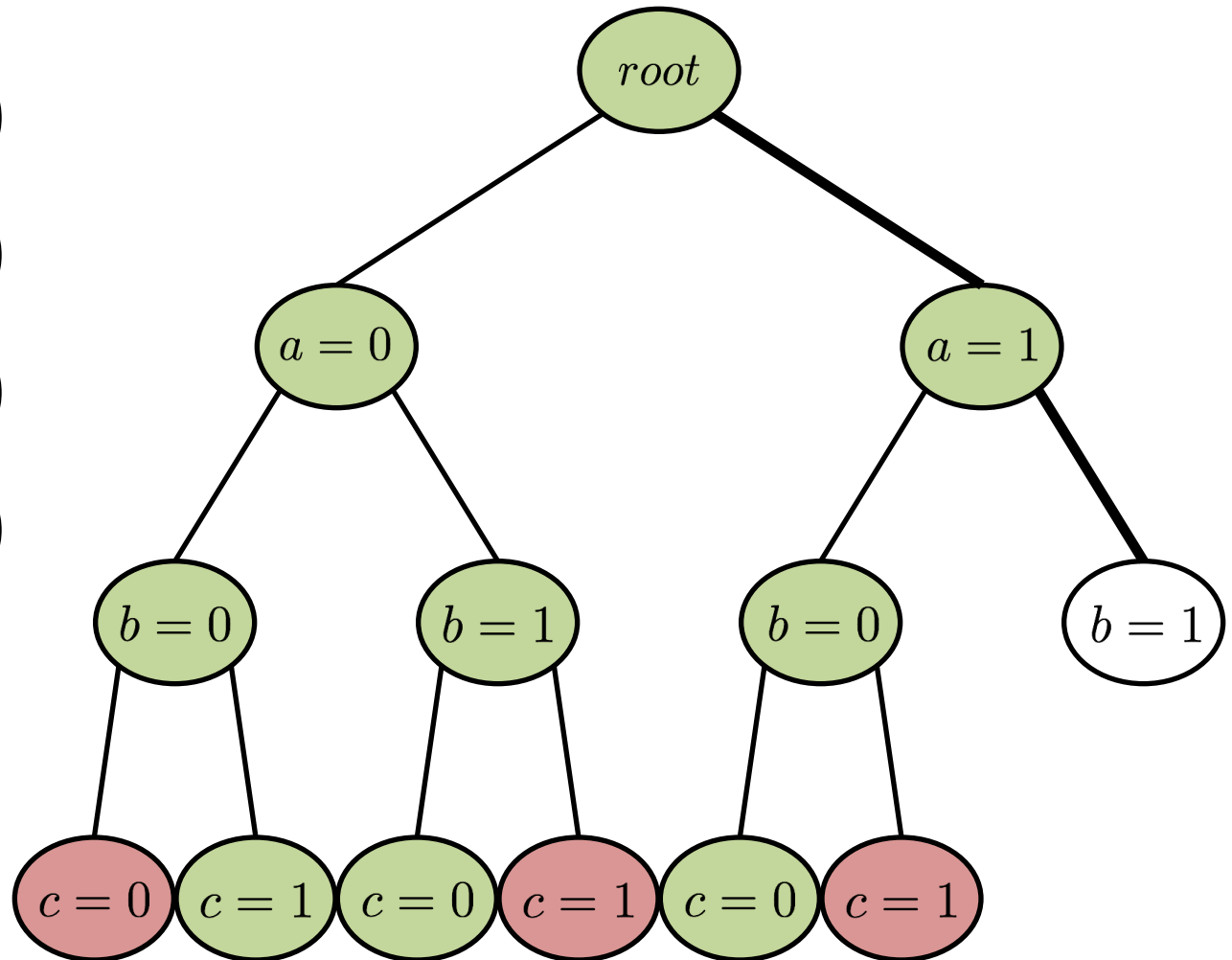
Satisfiable



Systematic Search with Backtracking

$$\begin{aligned} &(\underline{a} \vee \underline{b} \vee c) \\ &\wedge(\underline{\neg a} \vee \underline{\neg b}) \\ &\wedge(\underline{\neg b} \vee \neg c) \\ &\wedge(\neg c \vee \underline{\neg a}) \end{aligned}$$

Satisfiable



Systematic Search with Backtracking

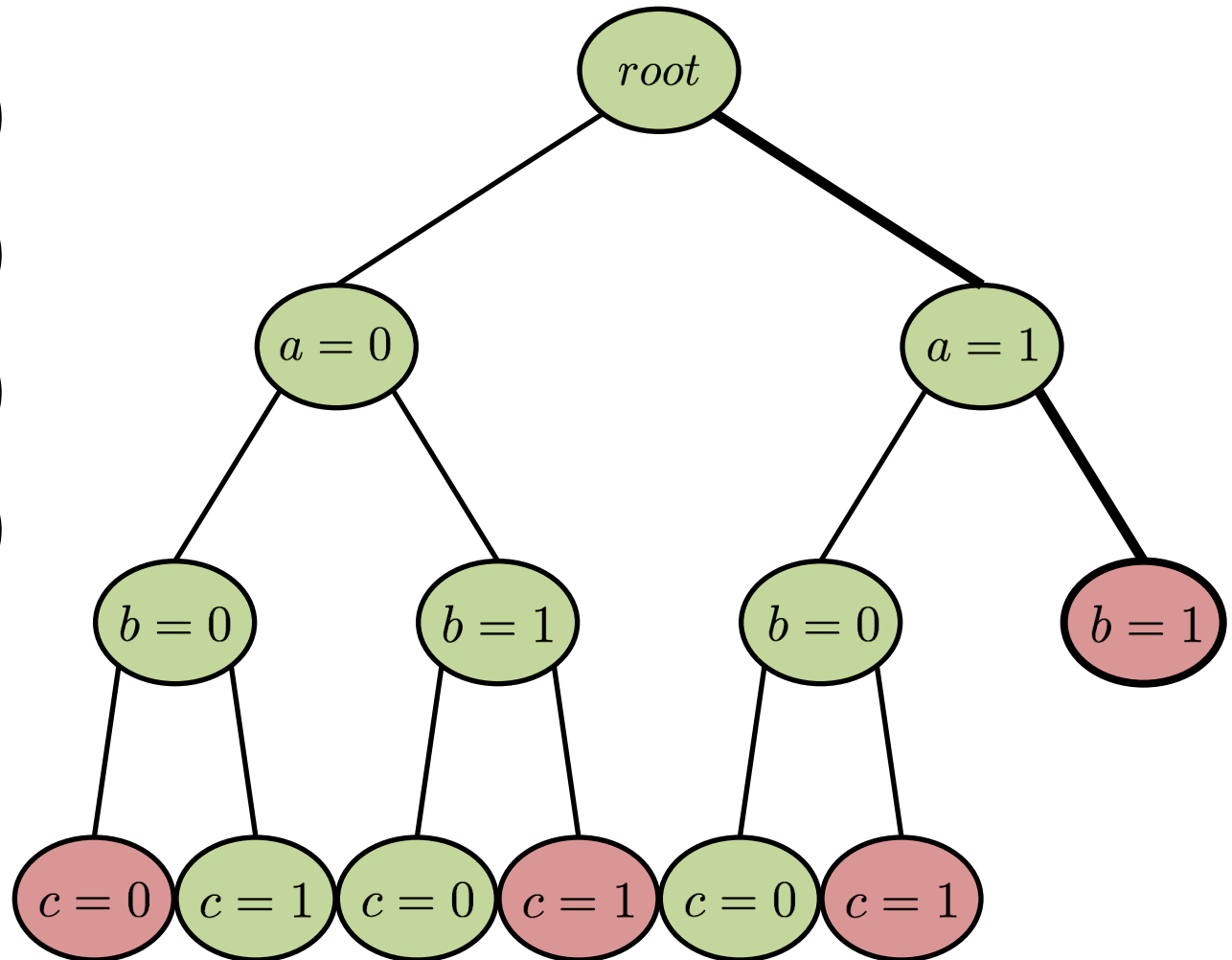
$$(\underline{a} \vee \underline{b} \vee c)$$

$$\wedge(\underline{\neg a} \vee \underline{\neg b})$$

$$\wedge(\underline{\neg b} \vee \neg c)$$

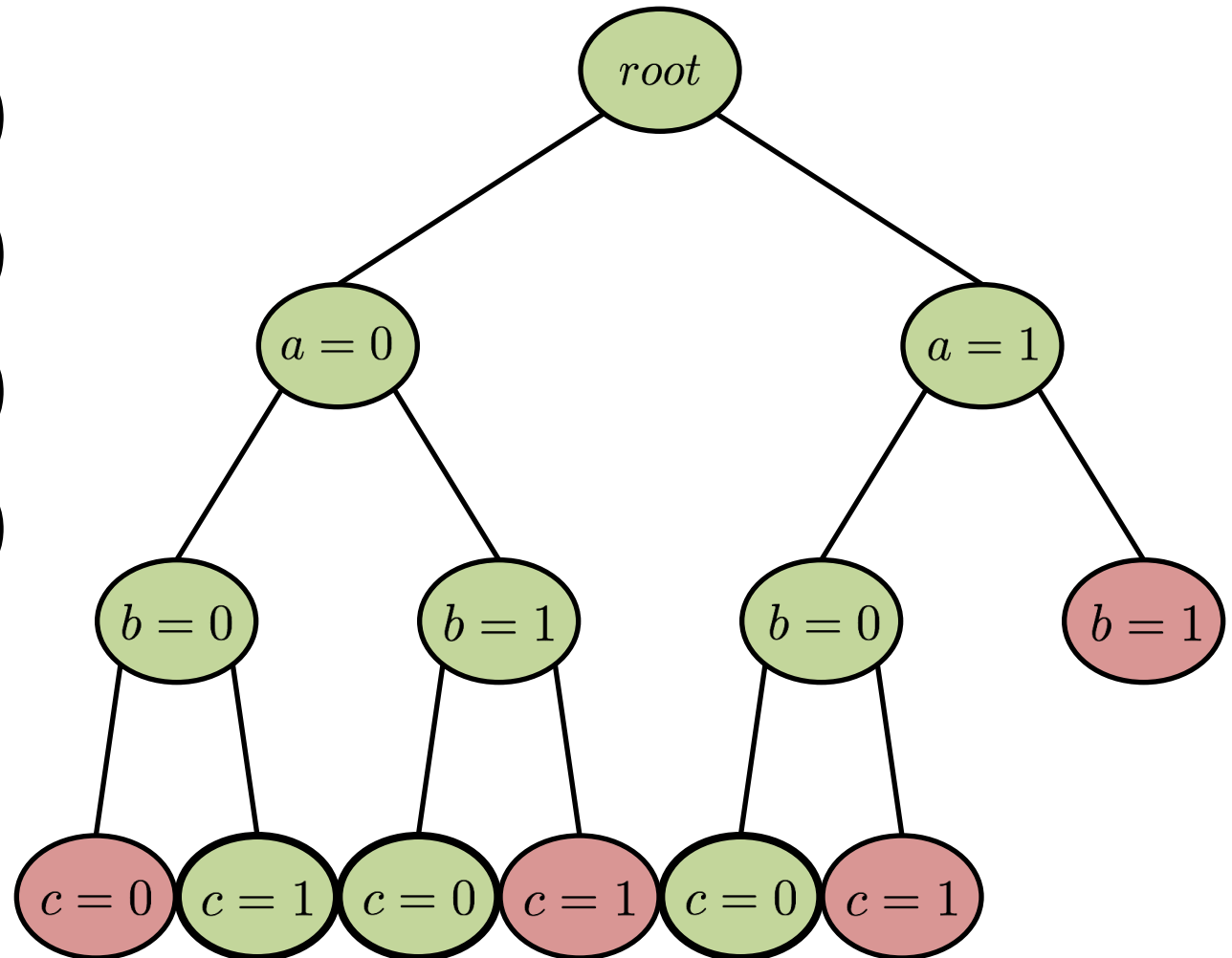
$$\wedge(\neg c \vee \underline{\neg a})$$

Satisfiable



Systematic Search with Backtracking

$$(a \vee b \vee c)$$
$$\wedge (\neg a \vee \neg b)$$
$$\wedge (\neg b \vee \neg c)$$
$$\wedge (\neg c \vee \neg a)$$



Satisfiable
3 solutions total