

Recitation 8

Created by Taylor Spangler, Adapted by Beau Christ

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- As a reminder, a few definitions:

1. **Reflexive:** $(a, a) \in R$ for all $a \in A$
2. **Symmetric:** $\forall a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$
3. **Antisymmetric:** $a, b \in A, (a, b) \in R$ and $(b, a) \in R$ then $a = b$
4. **Transitive:** $a, b, c \in A, (a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$
5. **Irreflexive:** $\forall a \in A, (a, a) \notin R$
6. **Asymmetric:** $(a, b) \in R$ then $(b, a) \notin R$
7. **Equivalence Relation:** A relation that is *reflexive, symmetric, and transitive*.
8. **Partial Ordering:** A relation \mathbf{R} on a set \mathbf{S} that is *reflexive, antisymmetric, and transitive*

- Rosen 9.4.25(c). Use Algorithm 1 (Given on page 603) to compute the transitive closure of the relation $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$ on the set $\{1, 2, 3, 4\}$.

- We note the matrix of a relation R^x resulting from the composing the relation R with itself x times: M_{R^x} , alternatively: $M_R^{[x]}$.
- We note the relations composition operator \circ and the matrix product operator \cdot , alternatively, \odot .

$$M_R = M_{R^1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_{R^1 \circ R^1} = M_{R^1} \cdot M_{R^1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R \circ R^2} = M_{R^2} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^4} = M_{R \circ R^3} = M_{R^3} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = M_{R^1} \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So it was already transitive.

- Rosen 9.4.27(a)

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

So the transitive closure contains all 16 pairs. Is this transitive? **Yes**

- 9.5.3 a) Is $\{(f, g) \mid f(1) = g(0) \text{ and } f(0) = g(1)\}$ an equivalence relation on the set of all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$?
 - So this relation, contains (f, g) , if $f(1) = g(0)$ and $f(0) = g(1)$.
 - So is this relation Reflexive? No, $f(0) = f(0)$ and $f(1) = f(1)$, this is only true when $f(0) = f(1)$. But there are other cases, so this is not reflexive.

- Is this relation Symmetric? Yes, if $f(1) = g(0)$ and $f(0) = g(1)$, then $g(1) = f(0)$ and $g(0) = f(1)$. So: if (f, g) then (g, f)
- Is this relation Transitive? Suppose we have (f, g) and (g, c) . Then $f(1) = g(0)$ and $f(0) = g(1)$ and $g(1) = c(0)$ and $g(0) = c(1)$. Therefore $f(1) = c(1)$ and $f(0) = c(0)$ so it is not necessarily true that $f(1) = c(0)$ and $f(0) = c(1)$. So no this is not transitive..
- 9.5.3 b) How about $\{(f, g) \mid f(1) = g(1) \text{ or } f(0) = g(0)\}$
 - Is it Reflexive? Yes if $f(1) = g(1)$, then $g(1) = f(1)$, same for $f(0)$ and $g(0)$
 - Is it Symmetric? Yes, again similar to last time.
 - Is it Transitive? No, suppose $f(1) = g(1)$ and $g(0) = c(0)$, but $f(0) \neq c(0)$ and $f(1) \neq c(1)$. Then we have (f, g) and (g, c) , but we don't have (f, c) .
- Equivalence Relations: 51 - Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last 4 bits.
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- Partial orderings, 9.6.1 a) $\{(0,0), (1,1), (2,2), (3,3)\}$
 - Yes, it is reflexive, antisymmetric, and transitive
- Partial orderings, 9.6.3 b) Let S be the set of all people in the world, and let aRb be the relation a is not taller than b . is this a partial ordering?
 - Reflexive: clearly a cannot be taller than a , so (a, a) .
 - Antisymmetric: Suppose we have (a, b) is it possible to have (b, a) if $b \neq a$? Yes, suppose a and b have the same height, but are different people.
 - Transitive: Yes, if a is not taller than b and b is not taller than c then a is not taller than c .
 - So no, it is not antisymmetric, therefore it is not a partial ordering.