# Modeling Sudoku as a CNF Formula

Spring 2018
CSCE 235H Introduction to Discrete Structures
URL: cse.unl.edu/~cse235h
All questions: Piazza

# Sudoku



**Rules**
- Each cell filled with a number 1…9
- All nine numbers appear in each
  - row,
  - column, and
  - 3x3 block

  (i.e., no duplicates)

# Sudoku

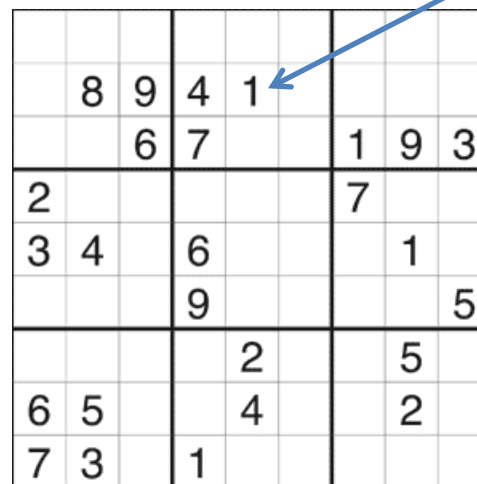| 1 | 7 | 3 | 2 | 6 | 9 | 5 | 8 | 4 |
|---|---|---|---|---|---|---|---|---|
| 5 | 8 | 9 | 4 | 1 | 3 | 6 | 7 | 2 |
| 4 | 2 | 6 | 7 | 5 | 8 | 1 | 9 | 3 |
| 2 | 9 | 1 | 5 | 8 | 4 | 7 | 3 | 6 |
| 3 | 4 | 5 | 6 | 7 | 2 | 8 | 1 | 9 |
| 8 | 6 | 7 | 9 | 3 | 1 | 2 | 4 | 5 |
| 9 | 1 | 4 | 8 | 2 | 6 | 3 | 5 | 7 |
| 6 | 5 | 8 | 3 | 4 | 7 | 9 | 2 | 1 |
| 7 | 3 | 2 | 1 | 9 | 5 | 4 | 6 | 8 |

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  – column, and
  – 3x3 block
  (i.e., no duplicates)

# Defining the Variables

- $p(i, j, n)$ asserts that
  - the cell in row $i$
  - column $j$
  - is assigned value $n$

$$p(2, 5, 1)$$

# Rules of the Game as Clauses

- Putting numbers in cells
  - Put a number in a given cell / Put a number in every cell
  - A given cell cannot take two numbers / No cell can take two numbers
- Constraining rows
  - Every number must appear in every row
- Constraining columns
  - Every number must appear in every column
- Constraining 3x3 blocks
  - Every number must appear in every 3x3 block

# Putting Numbers in Cells (1)
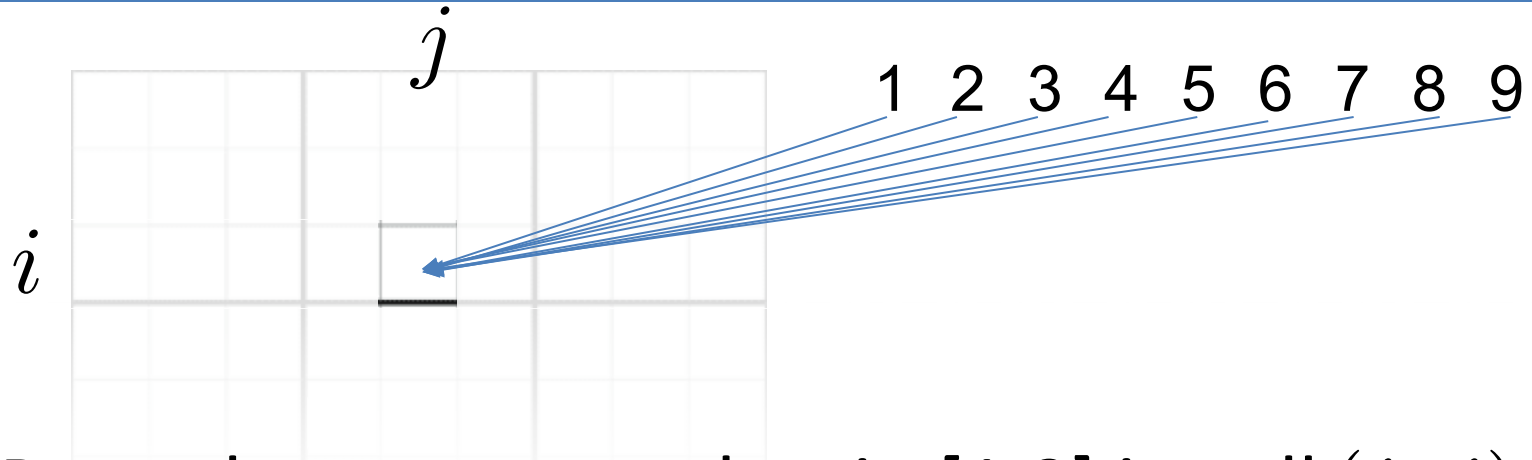


- Put at least one number in [1,9] in cell $(i, j)$

$$p(i, j, 1) \lor p(i, j, 2) \lor \ldots \lor p(i, j, 9)$$
$$\equiv \bigvee_{n=1}^{9} p(i, j, n)$$
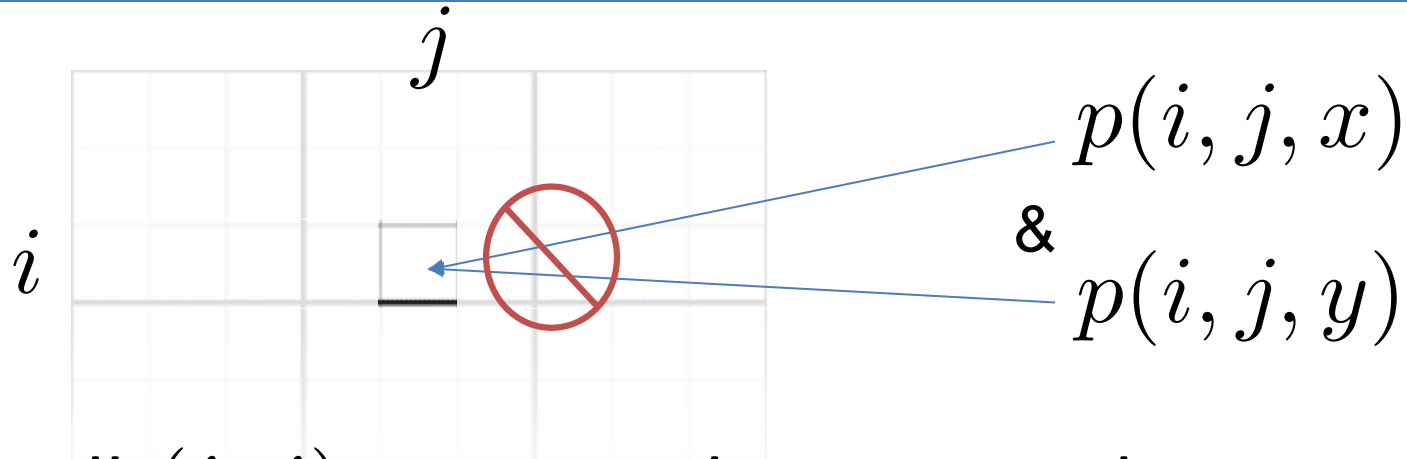
# Putting Numbers in Cells (2)

- Put a number in cell $(i, j)$

$$\bigvee_{n=1}^{9} p(i, j, n)$$

- Put a number in every cell

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigvee_{n=1}^{9} p(i, j, n)$$

Modeling Sudoku

# Putting Numbers in Cells (3)

$$j$$

$$p(i, j, x)$$

&

$$p(i, j, y)$$

$$i$$

- Cell $(i, j)$ cannot take two numbers

$\neg(p(i,j,1) \wedge (p,i,j,2)) \wedge \neg(p(i,j,1) \wedge (p,i,j,3)) \wedge \ldots \wedge \neg(p(i,j,1) \wedge (p,i,j,9)) \wedge$

$\neg(p(i,j,2) \wedge (p,i,j,3)) \wedge \neg(p(i,j,2) \wedge (p,i,j,4)) \wedge \ldots \wedge \neg(p(i,j,2) \wedge (p,i,j,9)) \wedge$

$$\forall x, y (x \neq y \rightarrow \neg(p(i,j,x) \wedge p(i,j,y)))$$

$$\equiv \forall x, y (x \neq y \rightarrow (\neg p(i,j,x) \vee \neg p(i,j,y)))$$

# Putting Numbers in Cells (4)

- Order of the literals does not matter so

$$\forall x, y(x \neq y \rightarrow ...) \quad \text{becomes} \quad \forall x, y(x < y \rightarrow ...)$$

$$x = 1, y = 2, 3, 4, 5, 6, 7, 8, 9$$
$$x = 2, y = 3, 4, 5, 6, 7, 8, 9$$
$$x = 3, y = 4, 5, 6, 7, 8, 9$$
$$x = 4, y = 5, 6, 7, 8, 9$$
$$x = 5, y = 6, 7, 8, 9$$
$$x = 6, y = 7, 8, 9$$
$$x = 7, y = 8, 9$$
$$x = 8, y = 9$$

$$\overset{8}{\underset{x=1}{\bigwedge}} \quad \overset{9}{\underset{y=x+1}{\bigwedge}} \quad (\neg p(i, j, x) \vee \neg p(i, j, y))$$
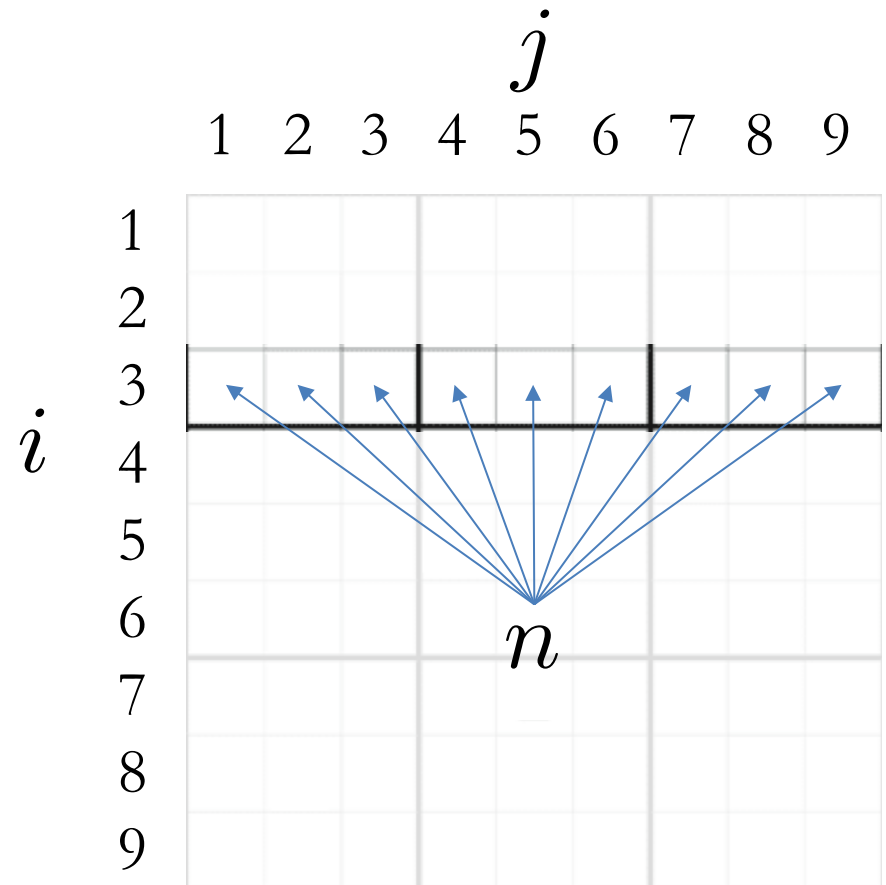
# Putting Numbers in cells (5)

- Every cell takes at most one number

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{x=1}^{8} \bigwedge_{y=x+1}^{9} (\neg p(i,j,x) \vee \neg p(i,j,y))$$

# Constraining Rows (1)

- For every
  - Row
    $i \in \{1,..,9\}$

  - Number
    $n \in \{1,..,9\}$

  - Select some column
    $j \in \{1,..,9\}$
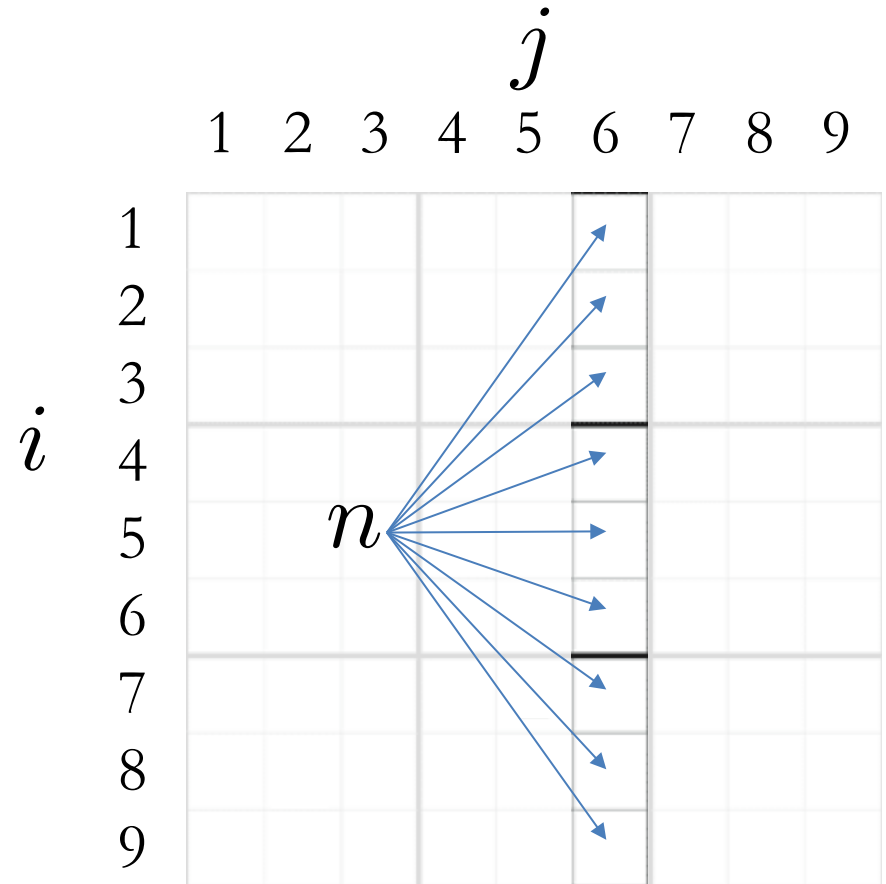
# Constraining Rows (2)

- Every number must appear in every row

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$

# Constraining Columns (1)

- For every
  - Column
    $j \in \{1, .., 9\}$

  - Number
    $n \in \{1, .., 9\}$

  - Select some row
    $i \in \{1, .., 9\}$
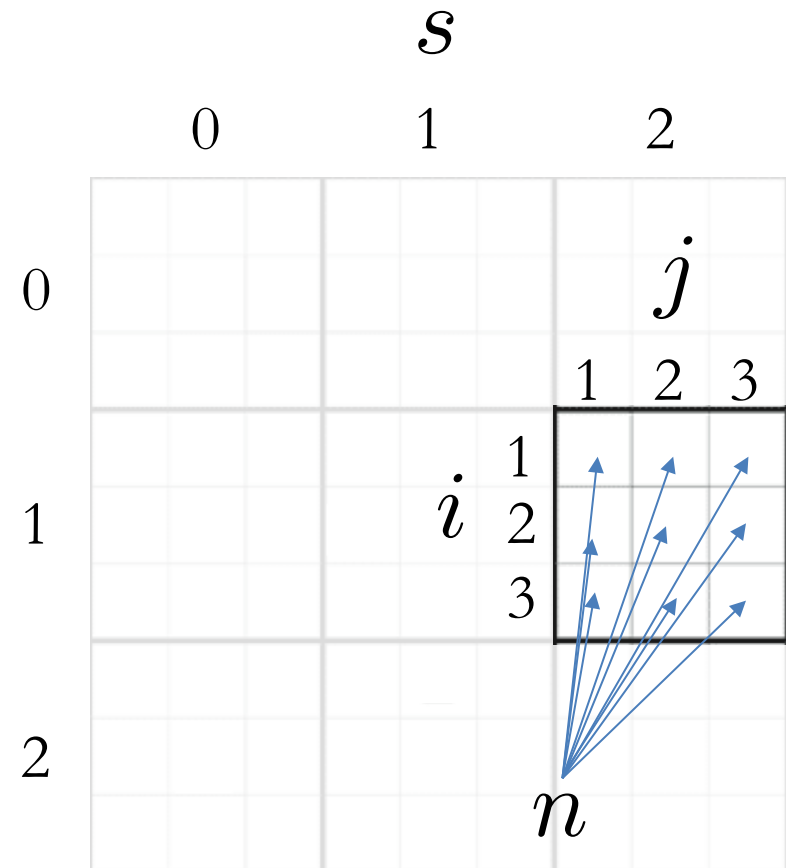
# Constraining Columns (2)

- Every number must appear in every column

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

# Constraining 3x3 Blocks (1)

- For every
  - Block
    $(r, s) \in \{0, 1, 2\} \times \{0, 1, 2\}$

  - Number
    $n \in \{1, .., 9\}$

  - Select some cell in block
    $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$

# Constraining 3x3 Blocks (2)

- Every number must appear in every 3x3 block

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

# Sudoku CNF Formula

- 729 variables

- 324 clauses with 9 literals

- 2916 clauses with 2 literals

- 3240 total clauses (+ clauses for initial setup)

# Redundant Clauses

- We can model a Sudoku as a SAT sentence in many ways

- Some models may have redundant clauses
  - Removing redundant clauses does not change the problem and yields an equivalent formula
  - Example: we can generate redundant clauses by applying inference rules on some clauses in the model

- Redundant clauses
  - Do not change the model
  - May speed the solving process.. Useful!

# Declaring the Hints/Clues

- We add propositions as unary clauses corresponding to hints/clues (filled cells)

$$\vdots$$
$$\wedge p(2, 2, 8)$$
$$\wedge p(2, 3, 9)$$
$$\wedge p(2, 4, 4)$$
$$\wedge p(2, 5, 1)$$
$$\wedge p(3, 3, 6)$$
$$\wedge \ldots$$

# Solving Sudoku

- Total number of possible assignments

$$2^{729} = 2.824014 \times 10^{219}$$

- Testing one billion assignments a second

$$8.94876 \times 10^{202} \text{ years}$$

- SAT solvers solve a Sudoku in milliseconds by aggressively pruning the search tree