Relations

Section 9.1, 9.3—9.5 of Rosen

Spring 2018

CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

Outline

- Relation:
  - Definition, representation, relation on a set
- Properties
  - Reflexivity, symmetry, antisymmetric, irreflexive, asymmetric
- Combining relations
  - $\cap$, $\cup$, $\backslash$, composite of relations
- Representing relations
  - 0-1 matrices, directed graphs
- Closure of relations
  - Reflexive closure, diagonal relation, Warshall's Algorithm,
- Equivalence relations:
  - Equivalence class, partitions,

Introduction

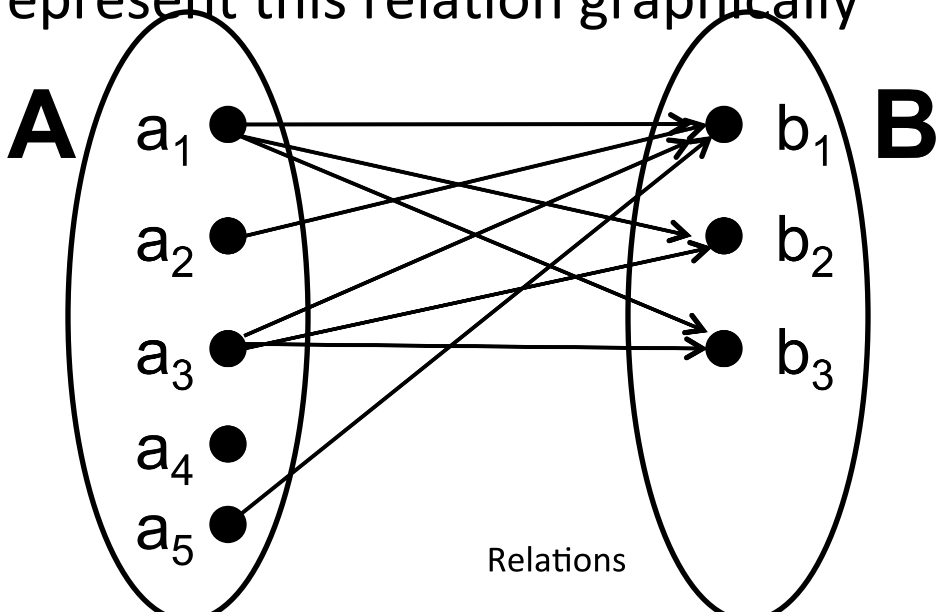
- A relation between elements of two sets is a subset of their Cartesian products (set of all ordered pairs
- **Definition**: A binary relation from a set A to a set B is a subset $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B \}$
- Relation versus function
  - In a relation, each $a \in A$ can map to multiple elements in B
  - Relations are more general than functions
- When $(a,b) \in R$, we say that a is <u>related</u> to b.
- Notation: aRb, $a \not R b$ aRb, $a\notR b$

Relations: Representation

- To represent a relation, we can enumerate every element of R
- Example
  - Let A={a_1,a_2,a_3,a_4,a_5} and B={b_1,b_2,b_3}
  - Let *R* be a relation from A to B defined as follows

$$R=\{(a_1,b_1),(a_1,b_2),(a_1,b_3), (a_2,b_1),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_5,b_1)\}$$

- We can represent this relation graphically



Graphical representation
- *Bipartite*
- *Directed*
- *Graph*

Relations on a Set

- **Definition**: A relation on the set A is a relation from A to A and is a subset of $A \times A$
- **Example**

  The following are binary relations on N

$$R_1=\{ (a,b) \mid a \leq b \}$$

$$R_2=\{ (a,b) \mid a,b \in N,\ a/b \in Z \}$$

$$R_3=\{ (a,b) \mid a,b \in N, a-b=2 \}$$

- **Question**

  For each of the above relations, give some examples of ordered pairs $(a,b) \in N^2$ that are not in the relation

Properties

- We will study several properties of relations
  - Reflexive
  - Symmetric
  - Transitive
  - Antisymmetric
  - Asymmetric
- Alert: Those properties are defined for only relations on a set

Properties: Reflexivity

- In a relation on a set, if all ordered pairs (a,a) for every $a \in A$ appears in the relation, R is called reflexive
- **Definition**: A relation *R* on a set A is called **reflexive** *iff*

$$\forall a \in A \ (a,a) \in R$$

Reflexivity: Examples

- Recall the relations below, which is reflexive?

R_1={ (a,b) | a ≤ b }

R_2={ (a,b) | a,b∈*N*, a/b∈*Z* }

R_3={ (a,b) | a,b∈*N*, a-b=2 }

- R_1 is reflexive since for every a∈*N*, a ≤ a
- R_2 is reflexive since a/a=1 is an integer
- R_3 is not reflexive since a-a=0 for every a∈*N*

Properties: Symmetry

- **Definitions**
  - A relation R on a set A is called **symmetric** if

    $$\forall a,b \in A\ (\ (b,a) \in R \Leftrightarrow (a,b) \in R\ )$$

  - A relation R on a set A is called **antisymmetric** if

    $$\forall a,b \in A\ [\ (a,b) \in R \wedge (b,a) \in R \Rightarrow a=b]$$

Symmetry versus Antisymmetry

- In a symmetric relation aRb ⇔ bRa
- In an antisymmetric relation, if we have aRb and bRa hold only when a=b
- An antisymmetric relation is not necessarily a reflexive relation: it may be reflexive or not
- A relation can be
  - <u>both</u> symmetric and antisymmetric
  - or neither
  - or have one property but not the other

Symmetric Relations: Example

- Consider $R=\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2=1\}$, is R
  - Reflexive?
  - Symmetric?
  - Antisymmetric?
- R is not reflexive since for example $(2,2)\notin \mathbb{R}^2$
- R is symmetric because

  $\forall x,y\in \mathbb{R},\ xRy \Rightarrow x^2+y^2=1 \Rightarrow y^2+x^2=1 \Rightarrow yRx$
- R is not antisymmetric because $(1/3,\sqrt{8}/3)\in R$ and $(\sqrt{8}/3,1/3)\in R$ but $1/3\neq\sqrt{8}/3$

Properties: Transitivity

- **Definition**: A relation R on a set A is called **transitive**
  – if whenever $(a,b)\in R$ and $(b,c)\in R$
  – then $(a,c)\in R$ for all $a,b,c \in A$

$$\forall a,b,c \in A\ ((aRb) \wedge (bRc)) \Rightarrow aRc$$

Transitivity: Examples (1)

- Is the relation $R=\{(x,y)\in\mathbb{R}^2 \mid x\leq y\}$ transitive?

  Yes, it is transitive because xRy and $yRz \Rightarrow x\leq y$ and $y\leq z \Rightarrow x\leq z \Rightarrow xRz$

- Is the relation $R=\{(a,b),(b,a),(a,a)\}$ transitive?

  No, it is not transitive because bRa and aRb but $b\not{R}b$

Transitivity: Examples (2)

- Is the relation {(a,b) | a is an ancestor of b} transitive?

  Yes, it is transitive because a*R*b and b*R*c $\Rightarrow$ a is an ancestor of b and b is an ancestor of c $\Rightarrow$ a is an ancestor of c $\Rightarrow$ a*R*c

- Is the relation $\{(x,y)\in\mathbb{R}^2 \mid x^2 \geq y\}$ transitive?

  No, it is not transitive because $2R4$ and $4R10$ but $2\not R10$

More Properties

- **Definitions**
  - A relation on a set A is **irreflexive** iff $\forall a \in A\ (a,a) \notin R$
  - A relation on a set A is **asymmetric** iff

$$\forall a,b \in A\ (\ (a,b) \in R \Rightarrow (b,a) \notin R\ )$$

- **Lemma**: A relation R on a set A is **asymmetric** iff
  - R is irreflexive and
  - R is antisymmetric
- **Alert**

A relation that is not symmetric is not necessarily asymmetric

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- **Combining relations**
  - **$\cap$, $\cup$, \, composite of relations**
- Representing relations
  - 0-1 matrices, directed graphs
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Combining Relations

- Relations are simply… sets (of ordered pairs); subsets of the Cartesian product of two sets
- Therefore, in order to <u>combine</u> relations to create new relations, it makes sense to use the usual set operations
  - Intersection ($R_1 \cap R_2$)
  - Union ($R_1 \cup R_2$)
  - Set difference ($R_1 \setminus R_2$)
- Sometimes, combining relations endows them with the properties previously discussed. For example, two relations may be not transitive, but their union may be

Combining Relations: Example

- Let
  - $A=\{1,2,3,4\}$
  - $B=\{1,2,3,4\}$
  - $R_1=\{(1,2),(1,3),(1,4),(2,2),(3,4),(4,1),(4,2)\}$
  - $R_2=\{(1,1),(1,2),(1,3),(2,3)\}$
- Let
  - $R_1 \cup R_2=$
  - $R_1 \cap R_2 =$
  - $R_1 \setminus R_2 =$
  - $R_2 \setminus R_1 =$

Composite of Relations

- **Definition**: Let R_1 be a relation from the set A to B and R_2 be a relation from B to C, i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C$$

the <u>composite of</u> R_1 and R_2 is the relation consisting of ordered pairs (a,c) where $a \in A$, $c \in C$ and for which there exists an element $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. We denote the composite of R_1 and R_2 by

$$R_2 \circ R_1$$

Powers of Relations

- Using the composite way of combining relations (similar to function composition) allows us to recursively define power of a relation R on a set A
- **Definition**: Let R be a relation on A. The powers R^n, n=1,2,3,…, are defined recursively by

$$R^1 = R$$

$$R^{n+1} = R^n \circ R$$

Powers of Relations: Example

- Consider R={(1,1),(2,1),(3,2),(4,3)}
- R^2=
- R^3=
- R^4=
- Note that R^n=R^3 for n=4,5,6,…

Powers of Relations & Transitivity

- The powers of relations give us a nice characterization of transitivity
- **Theorem**: A relation R is transitive if and only if $R^n \subseteq R$ for n=1,2,3,…

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Representing Relations

- We have seen one way to <u>graphically</u> represent a function/relation between <u>two</u> (different) sets: Specifically as a directed graph with arrows between nodes that are related
- We will look at two alternative ways to represent relations
  - 0-1 matrices (bit matrices)
  - Directed graphs

0-1 Matrices (1)

- A 0-1 matrix is a matrix whose entries are 0 or 1
- Let *R* be a relation from A={a_1,a_2,…,a_n} and B={b_1,b_2, …,b_n}
- Let's impose an ordering on the elements in each set. Although this ordering is arbitrary, it is important that it remain consistent. That is, once we fix an ordering, we have to stick to it.
- When A=B, *R* is a relation on A and we choose the same ordering in the two dimensions of the matrix

0-1 Matrix (2)

- The relation R can be represented by a $(n\times m)$ sized 0-1 matrix $M_R=[m_{i,j}]$ as follows

$$m_{i,j} = \begin{cases} 1 \text{ if } (a_i,b_i) \in R \\ 0 \text{ if } (a_i,b_i) \notin R \end{cases}$$

- Intuitively, the (i,j)-th entry if 1 if and only if $a_i \in A$ is related to $b_i \in B$

0-1 Matrix (3)

- An important note: the choice of row-major or column-major form is important.
  - The $(i, j)^{\text{th}}$ entry refers to the i-th row &the j-th column.
  - The size, ($n \times m$), refers to the fact that M_R has n rows and m columns
- Though the choice is arbitrary, switching between row-major and column-major is a bad idea, because when $A \neq B$, the Cartesian Product $A \times B \neq B \times A$
- In matrix terms, the transpose, $(M_R)^T$ does not give the same relation. This point is moot for A=B.

0-1 Matrix (4)

$$
\begin{array}{c c c}
 & & \overbrace{\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array}}^{B} \\
A & \left\{\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}\right. & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

Matrix Representation: Example

- Consider again the example
  - $A=\{a_1,a_2,a_3,a_4,a_5\}$ and $B=\{b_1,b_2,b_3\}$
  - Let R be a relation from A to B as follows:

  $R=\{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_5,b_1)\}$
- Give M_R
  - What is the size of the matrix?

Using the Matrix Representation (1)

- A 0-1 matrix representation makes it very easy to check whether or not a relation is
  - Reflexive
  - Symmetric
  - Antisymmetric
- **Reflexivity**
  - For R to be reflexive, $\forall a\ (a,a) \in R$
  - In M_R, R is reflexive iff $m_{i,i}=1$ for $i=1,2,\ldots,n$
  - We check only the diagonal

Using the Matrix Representation (2)

- **Symmetry**
  - R is symmetric iff for all pairs (a,b) $aRb \Rightarrow bRa$
  - In M_R,   this is equivalent to $m_{i,j}=m_{j,i}$ for every pair i,j=1,2,…,n
  - We check that $M_R=(M_R)^T$
- **Antisymmetry**
  - R is antisymmetric if $m_{i,j}=1$ with $i\neq j$, then $m_{j,i}=0$
  - Thus, $\forall i,j=1,2,\ldots, n,\ i\neq j\ (m_{i,j}=0) \vee (m_{j,i}=0)$
  - A simpler logical equivalence is

$$\forall i,j=1,2,\ldots, n,\ i\neq j\ \neg((m_{i,j}=1) \wedge (m_{j,i}=1))$$

Matrix Representation: Example

- Is R reflexive? Symmetric? Antisymmetric?

$$\mathbf{M}_{R} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- Clearly R is not reflexive: $m_{2,2}=0$
- It is not symmetric because $m_{2,1}=1$, $m_{1,2}=0$
- It is however antisymmetric

Matrix Representation: Combining Relations

- Combining relations is also simple: union and intersection of relations are nothing more than entry-wise Boolean opertions
- **Union**: An entry in the matrix of the union of two relations $R_1 \cup R_2$ is 1 *iff* <u>at least one</u> of the corresponding entries in R_1 or R_2 is 1. Thus

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$$

- **Intersection**: An entry in the matrix of the intersection of two relations $R_1 \cap R_2$ is 1 *iff* <u>both</u> of the corresponding entries in R_1 and R_2 are 1. Thus

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$$

- Count the number of operations

Combining Relations: Example

- What is $M_{R1 \cup R2}$ and $M_{R1 \cap R2}$?

$$\mathbf{M}_{\boldsymbol{R1}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{M}_{\boldsymbol{R2}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{M}_{\boldsymbol{R1 \cup R2}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \mathbf{M}_{\boldsymbol{R1 \cap R2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- How does combining the relations change their properties?

Composing Relations: Example

- 0-1 matrices are also useful for composing matrices. If you have not seen matrix product before, read Section 3.8

$$\mathbf{M}_{R1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathbf{M}_{R2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{M}_{R2 \circ R1} = \mathbf{M}_{R1} \cdot \mathbf{M}_{R2} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Composite Relations: R^n

- Remember that recursively composing a relation $R^n \subseteq R$ for n=1,2,3,… gives a nice characterization of transitivity
- **Theorem**: A relation R is transitive if and only if $R^n \subseteq R$ for n=1,2,3,…
- We will use
  - this idea and
  - the composition by matrix multiplication

  to build the Warshall (a.k.a. Roy-Warshall) algorithm, which computed the <u>transitive closure</u> (discussed in the next section)

Directed Graphs Representation (1)

- We will study graphs in details towards the end of the semester
- We briefly introduce them here to use them to represent relations
- We have already seen directed graphs to represent functions and relations (between two sets). Those are special graphs, called <u>bipartite</u> directed graphs
- For a relation on a set A, it makes more sense to use a <u>general</u> directed graph rather than having two copies of the same set A

Definition: Directed Graphs (2)

- **Definition**: A *G* graph consists of
  - A set *V* of vertices (or nodes), and
  - A set *E* of edges (or arcs)
  - We note: $G=(V,E)$
- **Definition**: A directed *G* graph (digraph) consists of
  - A set *V* of vertices (or nodes), and
  - A set *E* of edges of ordered pairs of elements of V (of vertices)

Directed Graphs Representation (2)

- **Example**:
  - Let $A = \{a_1, a_2, a_3, a_4\}$
  - Let R be a relation on A defined as follows

$$R=\{(a_1,a_2),(a_1,a_3),(a_1,a_4),(a_2,a_3),(a_2,a_4),(a_3,a_1),(a_3,a_4),(a_4,a_3),(a_4,a_4)\}$$

- Draw the digraph representing this relation (see white board)

Using the Digraphs Representation (1)

- A directed graph offers some <u>insight</u> into the properties of a relation
- **Reflexivity**: In a digraph, the represented relation is reflexive *iff* every vertex has a self loop
- **Symmetry**: In a digraph, the represented relation is symmetric *iff* for every directed edge from a vertex x to a vertex y there is also an edge from y to x

Using the Digraphs Representation (2)

- **Antisymmetry**: A represented relation is antisymmetric *iff* there is never a back edge for any directed edges between two distinct vertices
- **Transitivity**: A digraph is transitive if for every pair of directed edges (x,y) and (y,z) there is also a directed edge (x,z)
  - → This may be harder to visually verify in more complex graphs

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Closures: Definitions

- If a given relation R
  - is not reflexive (or symmetric, antisymmetric, transitive)
  - How can we transform it into a relation R' that is?
- **Example**: Let $R=\{(1,2),(2,1),(2,2),(3,1),(3,3)\}$
  - How can we make it reflexive?
  - In general we would like to change the relation as little as possible
  - To make R reflexive, we simply add (1,1) to the set
- **Inducing a property on a relation is called its closure.**
- Above, $R' = R \cup \{(1,1)\}$ is called the reflexive closure

Reflexive Closure

- In general, the reflexive closure of a relation R on A is $R \cup \Delta$ where $\Delta = \{ (a,a) \mid a \in A \}$
- Δ is the diagonal relation on A
- **Question**: How can we compute the diagonal relation using
  - 0-1 matrix representation?
  - Digraph representation?

Symmetric Closure

- Similarly, we can create the <u>symmetric closure</u> using the inverse of the relation R.
- The symmetric closer is, $R \cup R'$ where

$$R' = \{ (b,a) \mid (a,b) \in R \}$$

- **Question**: How can we compute the symmetric closure using
  – 0-1 matrix representation?
  – Digraph representation?

Transitive Closure

- To compute the transitive closure we use the theorem
- **Theorem**: A relation R is transitive if and only if $R^n \subseteq R$ for n=1,2,3,…
- Thus, if we compute R^k such that $R^\mathrm{k} \subseteq R^\mathrm{n}$ for all n≥k, then R^k is the transitive closure
- The Warshall's Algorithm allows us to do this efficiently
- **Note**: Your textbook gives much greater details in terms of graphs and connectivity relations. It is good to read this material, but it is based on material that we have not yet seen.

Warshall's Algorithm: Key Ideas

- In any set A with |A|=n, any transitive relation will be built from a sequence of relations that has a length of at most n. Why?
- Consider the case where the relation R on A has the ordered pairs $(a_1,a_2),(a_2,a_3),\ldots,(a_{n-1},a_n)$. Then, (a_1,a_n) must be in R for R to be transitive
- Thus, by the previous theorem, it suffices to compute (at most) R^n
- Recall that $R^k = R \circ R^{k-1}$ is computed using a bit-matrix product
- The above gives us a natural algorithm for computing the transitive closure: the Warshall's Algorithm

Warshall's Algorithm

Input: An (n×n) 0-1 matrix M_R representing a relation R on A, |A|=n

Output: An (n×n) 0-1 matrix W representing the transitive closure of R on A

1. $W \leftarrow M_R$
2. FOR k=1,…, n DO
3.   FOR i=1,…,n DO
4.   FOR j=1,…,n DO
5. $w_{i,j} \leftarrow w_{i.j} \vee (w_{i,k} \wedge w_{k,j})$
6.   END
7.   END
8. END
9. RETURN W

Warshall's Algorithm: Example

- Compute the transitive closure of
  - The relation R={(1,1),(1,2),(1,4),(2,2),(2,3),(3,1),(3,4),(4,1),(4,4)}
  - On the set A={1,2,3,4}

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Equivalence Relation

- Consider the set of every person in the world
- Now consider a *R* relation such that (a,b)∈*R* if a and b are siblings.
- Clearly this relation is
  - Reflexive
  - Symmetric, and
  - Transitive
- Such as relation is called an <u>equivalence relation</u>
- **Definition**: A relation on a set A is an equivalence relation if it is reflexive, symmetric, and transitive

Equivalence Class (1)

- Although a relation *R* on a set A may not be an equivalence relation, we can define a subset of A such that *R* does become an equivalence relation (on the subset)
- **Definition**: Let *R* be an equivalence relation on a set A and let $a \in A$. The set of all elements in A that are related to a is called the equivalence class of a. We denote this set $[a]_R$. We omit *R* when there is not ambiguity as to the relation.

$$[a]_R = \{ s \mid (a,s) \in R,\ s \in A \}$$

Equivalence Class (2)

- The elements in $[a]_R$ are called <u>representatives</u> of the equivalence class
- **Theorem**: Let *R* be an equivalence class on a set A. The following statements are equivalent
  - a*R*b
  - $[a]=[b]$
  - $[a] \cap [b] \neq \emptyset$
- The proof in the book is a circular proof

Partitions (1)

- Equivalence classes partition the set A into disjoint, non-empty subsets A_1, A_2, …, A_k
- A **partition** of a set A satisfies the properties
  - $\bigcup^{k}_{i=1} A_i = A$
  - $A_i \cap A_j = \emptyset$ for $i \neq j$
  - $A_i \neq \emptyset$ for all i

Partitions (2)

- **Example**: Let R be a relation such that $(a,b) \in R$ if a and b live in the same state, then R is an equivalence relation that partitions the set of people who live in the US into 50 equivalence classes
- **Theorem**:
  - Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.
  - Conversely, given a partition A_i of the set S, there is a equivalence relation R that has the set A_i as its equivalence classes

Partitions: Visual Interpretation

- **In a 0-1 matrix**, if the elements are ordered into their equivalence classes, equivalence classes/partitions form perfect squares of 1s (with 0s everywhere else)
- **In a diargh**, equivalence classes form a collections of disjoint <u>complete</u> graphs
- **Example**: Let A={1,2,3,4,5,6,7} and *R* be an equivalence relation that partitions A into A_1={1,2}, A_2={3,4,5,6} and A_3={7}
  - Draw the 0-1 matrix
  - Draw the digraph

Equivalence Relations: Example 1

- **Example**: Let R={ (a,b) | a,b∈$\mathbb{R}$ and a≤b}
  - Is R reflexive?
  - Is it transitive?
  - Is it symmetric?

    No, it is not. 4 is related to 5 (4 ≤ 5) but 5 is not related to 4

Thus R is <u>not</u> an equivalence relation

Equivalence Relations: Example 2

- **Example**: Let R={ (a,b) | a,b∈Z and a=b}
  - Is R reflexive?
  - Is it transitive?
  - Is it symmetric?
  - What are the equivalence classes that partition Z?

Equivalence Relations: Example 3

- **Example**: For (x,y),(u,v) $\in \mathbb{R}^2$, we define

$$R = \{ ((x,y),(u,v)) \mid x^2+y^2=u^2+v^2 \}$$

- Show that R is an equivalence relation.
- What are the equivalence classes that R defines (i.e., what are the partitions of $\mathbb{R}^2$)?

Equivalence Relations: Example 4

- **Example**: Given n,r$\in N$, define the set

$$nZ + r = \{ na + r \mid a \in Z \}$$

  - For n=2, r=0, 2Z represents the equivalence class of all even integers
  - What n, r give the class of all <u>odd</u> integers?
  - For n=3, r=0, 3Z represents the equivalence class of all integers divisible by 3
  - For n=3, r=1, 3Z represents the equivalence class of all integers divisible by 3 with a <u>remainder</u> of 1
  - In general, this relation defines equivalence classes that are, in fact, <u>congruence classes</u> (See Section 3.4)