Partial Orders

Section 9.6 of Rosen

Spring 2018

CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

Outline

- Motivating example
- Definitions
  - Partial ordering, comparability, total ordering, well ordering
- Principle of well-ordered induction
- Lexicographic orderings
- Hasse Diagrams
- Extremal elements
- Lattices
- Topological Sorting

Motivating Example (1)

- Consider the renovation of Avery Hall. In this process several tasks were undertaken
  - Remove Asbestos
  - Replace windows
  - Paint walls
  - Refinish floors
  - Assign offices
  - Move in office furniture

Motivating Example (2)

- Clearly, some things had to be done before others could begin
  - Asbestos had to be removed before anything (except assigning offices)
  - Painting walls had to be done before refinishing floors to avoid ruining them, etc.
- On the other hand, several things could be done concurrently:
  - Painting could be done while replacing the windows
  - Assigning offices could be done at anytime before moving in office furniture
- This scenario can be nicely modeled using <u>partial orderings</u>

Partial Orderings: Definitions

- **Definitions**:
  - A relation *R* on a set *S* is called a partial order if it is
    - Reflexive
    - Antisymmetric
    - Transitive
  - A set S together with a partial ordering R is called a partially ordered set (poset, for short) and is denote (*S*,*R*)
- Partial orderings are used to give an order to sets that may not have a natural one
- In our renovation example, we could define an ordering such that $(a,b) \in R$ if 'a must be done before b can be done'

Partial Orderings: Notation

- We use the notation:
  - $a \preccurlyeq b$, when $(a,b) \in R$ $\preccurlyeq$preccurlyeq$
  - $a \prec b$, when $(a,b) \in R$ and $a \neq b$ $\prec$prec$
- The notation $\prec$ is not to be mistaken for "less than" ($\preccurlyeq$ versus $\leq$)
- The notation $\prec$ is used to denote <u>any</u> partial ordering

Comparability: Definition

- **Definition**:
  - The elements a and b of a poset $(S, \preccurlyeq)$ are called <u>comparable</u> if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
  - When for $a,b \in S$, we have neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, we say that a,b are <u>incomparable</u>
- Consider again our renovation example
  - Remove Asbestos $\prec a_i$ for all activities a_i except assign offices
  - Paint walls $\prec$ Refinish floors
  - Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices

Total orders: Definition

- **Definition**:
  - If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a <u>totally ordered set</u>.
  - The relation $\preccurlyeq$ is said to be a <u>total order</u>
- Example
  - The relation "less than or equal to" over the set of integers $(Z, \leq)$ since for every $a,b \in Z$, it must be the case that $a \leq b$ or $b \leq a$
  - What happens if we replace $\leq$ with $<$?

    The relation $<$ is not reflexive, and $(Z,<)$ is not a poset

Well Orderings: Definition

- **Definition**: $(S,\preccurlyeq)$ is a well-ordered set if
  - It is a **poset**
  - Such that $\preccurlyeq$ is a total ordering and
  - Such that **every** non-empty subset of S has a least element
- Example
  - The natural numbers along with $\leq$, $(N,\leq)$, is a well-ordered set since any nonempty subset of N has a least element and $\leq$ is a total ordering on N
  - However, $(Z,\leq)$ is not a well-ordered set
    - Why? $Z^- \subset Z$ but does not have a least element
    - Is it totally ordered? Yes

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Principle of Well-Ordered Induction

- Well-ordered sets are the basis of the proof technique known as induction (more when we cover Chapter 3)
- **Theorem: Principle of Well-Ordered Induction**

  Given S is a well-ordered set. P(x) is true for all $x \in S$ if

  (**Basis Step**: $P(x_0)$ is true for the least element in S and)

  Inductive Step: For every $y \in S$ if P(x) is true for all $x \prec y$, then P(y) is true

Principle of Well-Ordered Induction: Proof

Proof: (S well ordered) $\wedge$(Basis Step) $\wedge$ (Induction Step) $\Rightarrow$ $\forall x \in S$, P(x)

- Suppose that it is not the case the P(x) holds for all $x \in S$

  $\Rightarrow \exists y$ P(y) is false

  $\Rightarrow$ A={ $x \in S$ | P(x) is false } is not empty
- S is well ordered $\Rightarrow$ A has a least element a
- Since $P(x_0)$ is true and P(a) is false $\Rightarrow$ $a \neq x_0$
- P(x) holds for all $x \in S$ and $x \prec a$, then P(a) holds by the induction step
- This yields a contradiction **QED**

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- **Lexicographic orderings**
  - **Idea, on $A_1 \times A_2$, $A_1 \times A_2 \times \ldots \times A_n$, S^t (strings)**
- Hasse Diagrams
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Lexicographic Orderings: Idea

- Lexigraphic ordering is the same as any dictionary or phone-book ordering:
  - We use alphabetic ordering
    - Starting with the first character in the string
    - Then the next character, if the first was equal, etc.
  - If a word is shorter than the other, than we consider that the ‘no character’ of the shorter word to be less than ‘a’

Lexicographic Orderings on $A_1 \times A_2$

- Formally, lexicographic ordering is defined by <u>two</u> other orderings
- **Definition**: Let $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$ be two posets. The <u>lexicographic ordering</u> $\prec$ on the Cartesian product $A_1 \times A_2$ is defined by

  $$(a_1,a_2) \prec (a'_1,a'_2) \text{ if } (a_1 \prec_1 a'_1) \text{ or } (a_1 = a'_1 \text{ and } a_2 \prec_2 a'_2)$$
- If we add equality to the lexicographic ordering $\prec$ on $A_1 \times A_2$, we obtain a partial ordering

Lexicographic Ordering on $A_1 \times A_2 \times \ldots \times A_n$

- Lexicographic ordering generalizes to the Cartesian Product of n set in a natural way
- Define $\preccurlyeq$ on $A_1 \times A_2 \times \ldots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

If a1 $\prec$ b1 or of there is an integer i>0 such that

$$a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_i = b_i \text{ and } a_{i+1} \prec b_{i+1}$$

Lexicographic Ordering on Strings

- Consider the two non-equal strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a poset $(S^t, \preccurlyeq)$
- Let
  - $t=\min(n,m)$
  - $\prec$ be the lexicographic ordering on S^t
- $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if
  - $(a_1,a_2,\ldots,a_t) \prec (b_1,b_2,\ldots,b_t)$ or
  - $(a_1,a_2,\ldots,a_t)=(b_1,b_2,\ldots,b_t)$ and $m<n$

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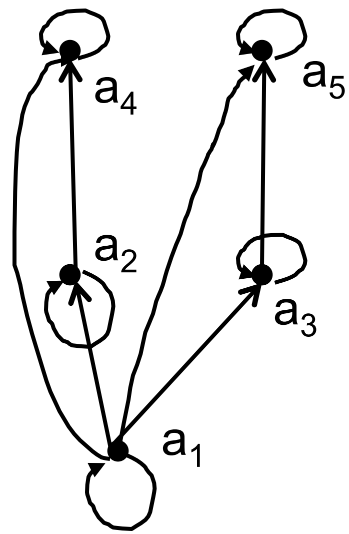
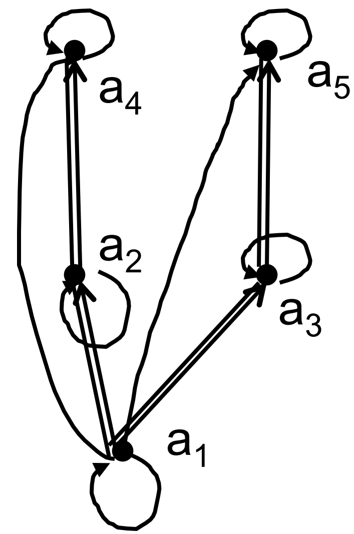
Hasse Diagrams

- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  - Consider the digraph representation of a partial order
  - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  - Thus, we can simplify the graph as follows
    - Remove all self loops
    - Remove all transitive edges
    - Remove directions on edges assuming that they are oriented upwards
  - The resulting diagram is far simpler

Hasse Diagram: Example

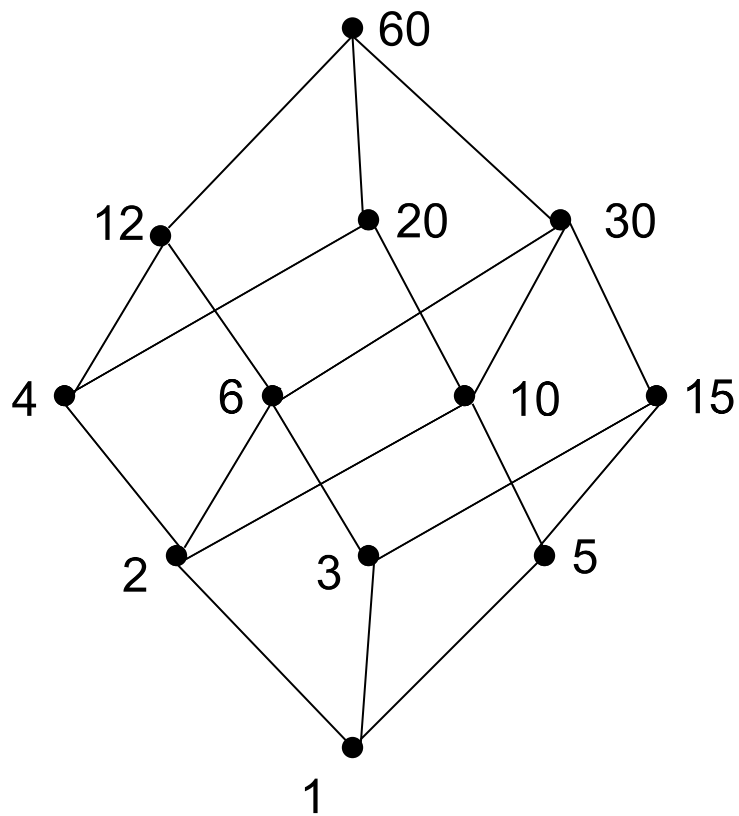




Hasse Diagrams: Example (1)

- Of course, you need not always start with the complete relation in the partial order and then trim everything.
- Rather, you can build a Hasse Diagram directly from the partial order
- Example: Draw the Hasse Diagram
  - for the following partial ordering: $\{(a,b) \mid a|b\}$
  - on the set $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
  - (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

Hasse Diagram: Example (2)



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Extremal Elements: Summary

We will define the following terms:

- A maximal/minimal element in a poset $(S, \preccurlyeq)$
- The maximum (greatest)/minimum (least) element of a poset $(S, \preccurlyeq)$
- An upper/lower bound element of a subset A of a poset $(S, \preccurlyeq)$
- The greatest lower/least upper bound element of a subset A of a poset $(S, \preccurlyeq)$

Extremal Elements: Maximal

- **Definition**: An element a in a poset (S, $\preccurlyeq$) is called maximal if it is not less than any other element in S. That is: $\neg(\exists b \in S\ (a \prec b))$
- If there is one unique maximal element a, we call it the maximum element (or the greatest element)

Extremal Elements: Minimal

- **Definition**: An element a in a poset $(S, \preccurlyeq)$ is called <u>minimal</u> if it is not greater than any other element in S. That is: $\neg(\exists b \in S\ (b \prec a))$
- If there is one <u>unique</u> minimal element a, we call it the <u>minimum</u> element (or the <u>least</u> element)

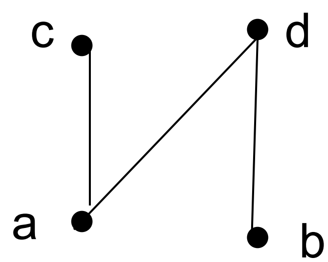
Extremal Elements: Upper Bound

- **Definition**: Let $(S,\preccurlyeq)$ be a poset and let $A \subseteq S$. If u is an element of S such that $a \preccurlyeq u$ for all $a \in A$ then u is an upper bound of A
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A. We abbreviate it as lub.

Extremal Elements: Lower Bound

- **Definition**: Let $(S,\preccurlyeq)$ be a poset and let $A \subseteq S$. If l is an element of S such that $l \preccurlyeq a$ for all $a \in A$ then l is an <u>lower bound of A</u>
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the <u>greatest lower bound on A</u>. We abbreviate it glb.

Extremal Elements: Example 1



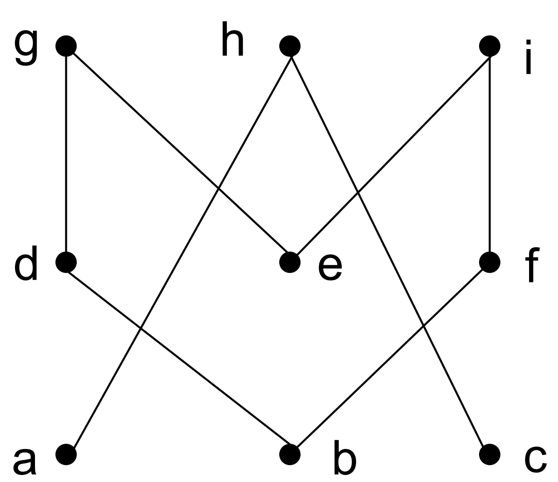
What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}



{d,e,f}

- Lower bounds: ∅, thus no glb
- Upper bounds: ∅, thus no lub

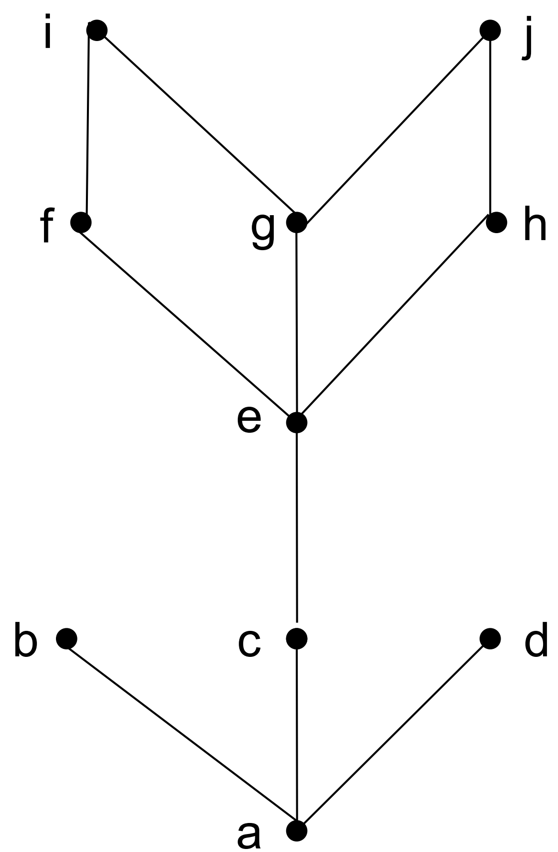
{a,c}

- Lower bounds: ∅, thus no glb
- Upper bounds: {h}, lub: h

{b,d}

- Lower bounds: {b}, glb: b
- Upper bounds: {d,g}, lub: d because $d \prec g$

Extremal Elements: Example 3



- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
  - Lower bounds: {a,c}, thus glb is c
  - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - Lower bounds: {a}, thus glb is a
  - Upper bounds: ∅, thus lub DNE

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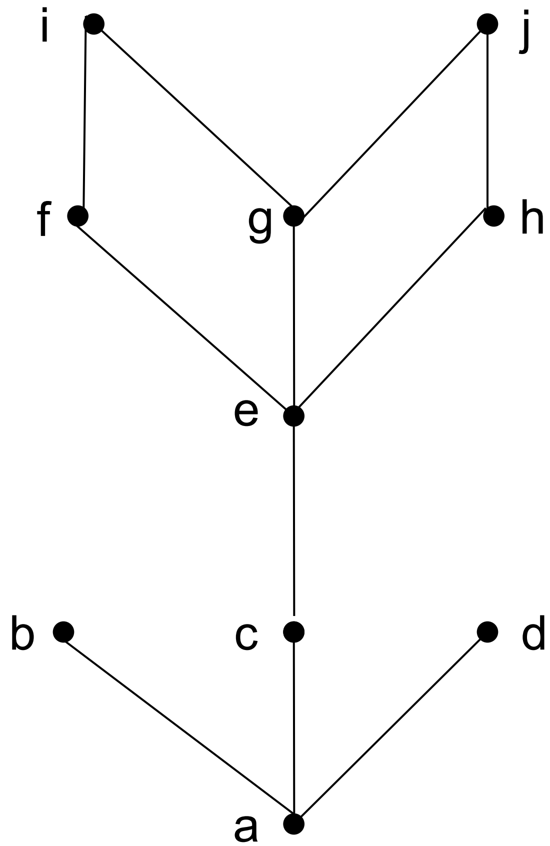
Lattices

- A special structure arises when every pair of elements in a poset has an lub and a glb
- **Definition**: A lattice is a partially ordered set in which every pair of elements has both
  - a least upper bound and
  - a greatest lower bound

Lattices: Example 1

- Is the example from before a lattice?

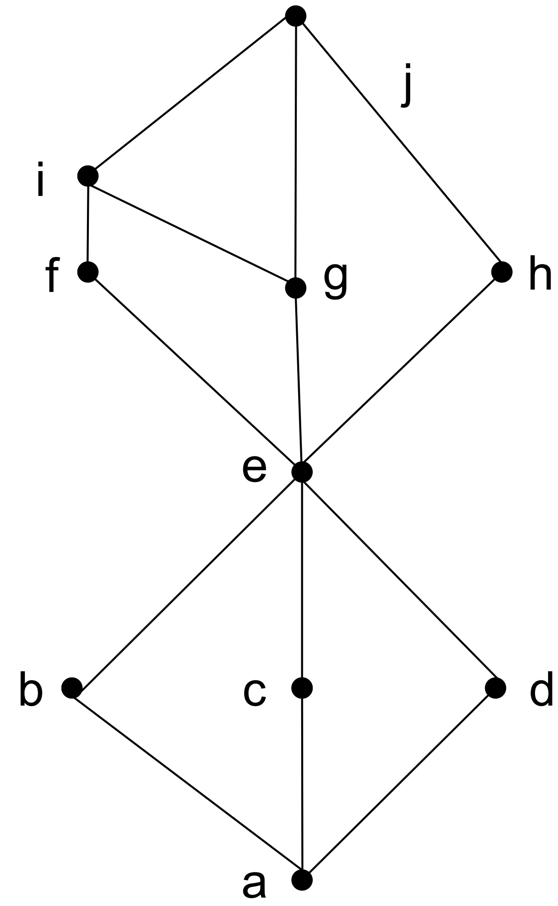
- **No, because the pair {b,c} does not have a least upper bound**



Lattices: Example 2

- What if we modified it as shown here?

- **Yes, because for any pair, there is an lub & a glb**



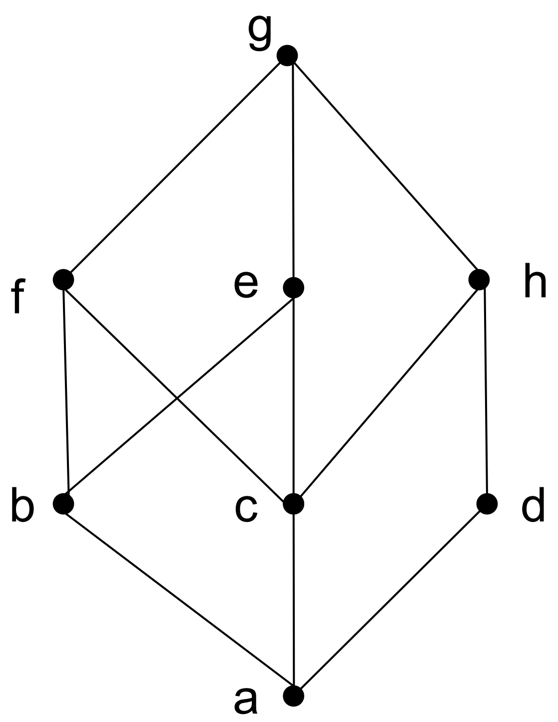
Lattices: Example 3

- Is this example a lattice?

No!

- The lower bound of A={e,f} is {a,b,c}
- However, A has no glb

Similarly, B={b,c} has no ulb



A Lattice Or Not a Lattice?

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be <u>incomparable</u> (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice

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Topological Sorting

- Let us return to the introductory example of Avery Hall renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule
- That is, given a <u>partial order</u>, we would like to transform it into a <u>total order</u> that is <u>compatible</u> with the partial order
- A total order is <u>compatible</u> if it does not violate any of the original relations in the partial order
- Essentially, we are simply <u>imposing an order on incomparable</u> elements in the partial order

Topological Sorting: Preliminaries (1)

- Before we give the algorithm, we need some tools to justify its correctness
- **Fact**: Every finite, nonempty poset $(S,\preccurlyeq)$ has a minimal element
- We will prove the above fact by a form of *reductio ad absurdum*

Topological Sorting: Preliminaries (2)

- **Proof**:
  - Assume, to the contrary, that a nonempty finite poset $(S,\prec)$ has no minimal element. In particular, assume that a_1 is not a minimal element.
  - Assume, w/o loss of generality, that $|S|=n$
  - If a_1 is not minimal, then there exists a_2 such that $a_2 \prec a_1$
  - But a_2 is also not minimal because of the above assumption
  - Therefore, there exists a_3 such that $a_3 \prec a_2$. This process proceeds until we have the last element a_n. Thus, $a_n \prec a_{n-1} \prec \ldots \prec a_2 \prec a_1$
  - Finally, by definition a_n is the minimal element **QED**

Topological Sorting: Intuition

- The idea of topological sorting is
  - We start with a poset $(S, \prec)$
  - We remove a minimal element, choosing arbitrarily if there is more than one. Such an element is guaranteed to exist by the previous fact
  - As we remove each minimal element, one at a time, the set S shrinks
  - Thus we are guaranteed that the algorithm will <u>terminate</u> in a finite number of steps
  - Furthermore, the order in which the elements are removed is a total order: $a_1 \prec a_2 \prec \ldots \prec a_{n-1} \prec a_n$
- Now, we can give the algorithm itself

Topological Sorting: Algorithm

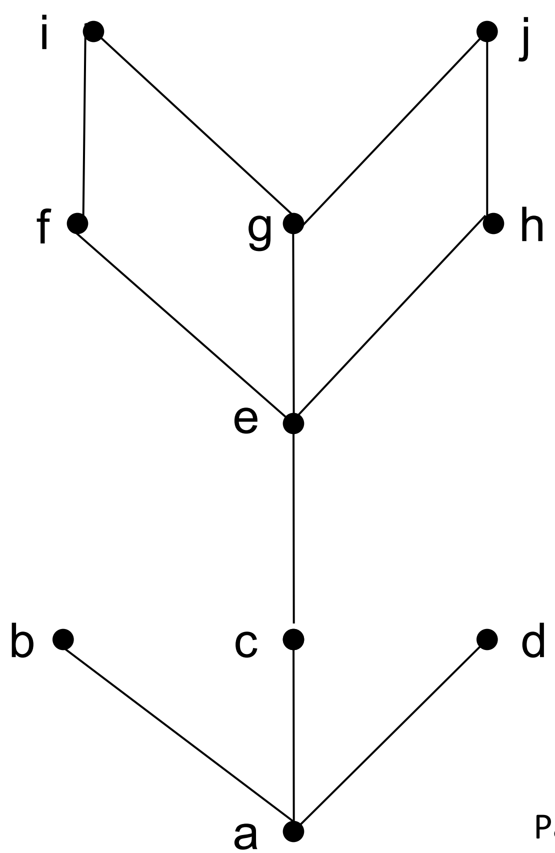
Input: $(S, \preccurlyeq)$ a poset with $|S|=n$

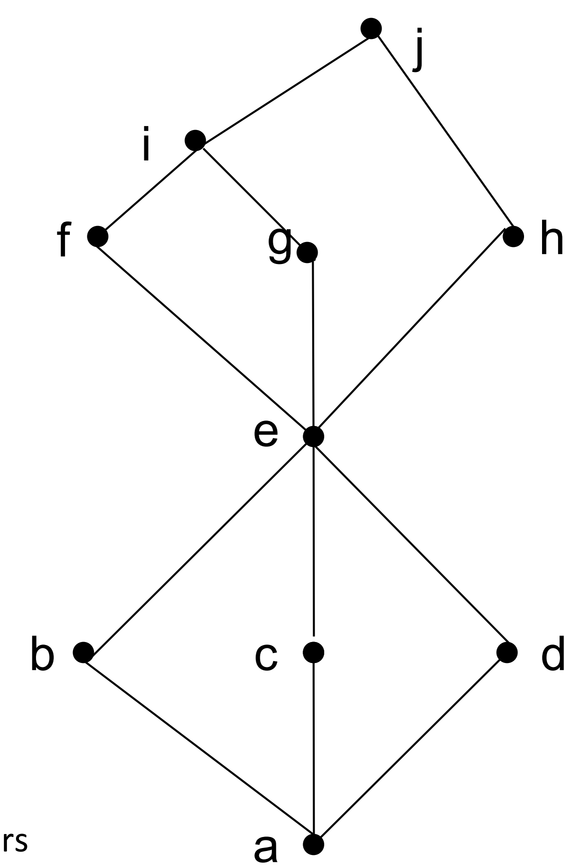
Output: A total ordering $(a_1, a_2, \ldots, a_n)$

1. $k \leftarrow 1$
2. **While** S **Do**
3. $\quad a_k \leftarrow$ a minimal element in S
4. $\quad S \leftarrow S \setminus \{a_k\}$
5. $\quad k \leftarrow k+1$
6. **End**
7. **Return** $(a_1, a_2, \ldots, a_n)$

Topological Sorting: Example

- Find a compatible ordering (topological ordering) of the poset represented by the Hasse diagrams below





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