

Master Theorem

Section 8.3 of Rosen

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CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

Outline

- Motivation
- The Master Theorem
 - Pitfalls
 - 3 examples
- 4th Condition
 - 1 example

Motivation: Asymptotic Behavior of Recursive Algorithms

- When analyzing algorithms, recall that we only care about the asymptotic behavior
- Recursive algorithms are no different
- Rather than solving exactly the recurrence relation associated with the cost of an algorithm, it is sufficient to give an asymptotic characterization
- The main tool for doing this is the master theorem

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Master Theorem

- Let $T(n)$ be a monotonically increasing function that satisfies

$$T(n) = a T(n/b) + f(n)$$

$$T(1) = c$$

where $a \geq 1$, $b \geq 2$, $c > 0$. If $f(n)$ is $\Theta(n^d)$ where $d \geq 0$ then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Master Theorem: Pitfalls

- You **cannot** use the Master Theorem if
 - $T(n)$ is not monotone, e.g. $T(n) = \sin(x)$
 - $f(n)$ is not a polynomial, e.g., $T(n) = 2T(n/2) + 2^n$
 - b cannot be expressed as a constant, e.g.

$$T(n) = T(\sqrt{n})$$

- Note that the Master Theorem does not solve the recurrence equation
- Does the base case remain a concern?

Master Theorem: Example 1

- Let $T(n) = T(n/2) + \frac{1}{2}n^2 + n$. What are the parameters?

$$a = 1$$

$$b = 2$$

$$d = 2$$

Therefore, which condition applies?

$1 < 2^2$, case 1 applies

- We conclude that

$$T(n) \in \Theta(n^d) = \Theta(n^2)$$

Master Theorem: Example 2

- Let $T(n) = 2T(n/4) + \sqrt{n} + 42$. What are the parameters?

$$a = 2$$

$$b = 4$$

$$d = 1/2$$

Therefore, which condition applies?

$$2 = 4^{1/2}, \text{ case 2 applies}$$

- We conclude that

$$T(n) \in \Theta(n^d \log n) = \Theta(\log n \sqrt{n})$$

Master Theorem: Example 3

- Let $T(n) = 3T(n/2) + 3/4n + 1$. What are the parameters?

$$a = 3$$

$$b = 2$$

$$d = 1$$

Therefore, which condition applies?

$3 > 2^1$, case 3 applies

- We conclude that

$$T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$$

- Note that $\log_2 3 \approx 1.584\dots$, can we say that $T(n) \in \Theta(n^{1.584})$

No, because $\log_2 3 \approx 1.5849\dots$ and $n^{1.584} \notin \Theta(n^{1.5849})$

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'Fourth' Condition

- Recall that we cannot use the Master Theorem if $f(n)$, the non-recursive cost, is not a polynomial
- There is a limited 4th condition of the Master Theorem that allows us to consider polylogarithmic functions
- **Corollary:** If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ for some $k \geq 0$ then
$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$$
- This final condition is fairly limited and we present it merely for sake of completeness.. Relax 😊

'Fourth' Condition: Example

- Say we have the following recurrence relation

$$T(n) = 2T(n/2) + n \log n$$

- Clearly, $a=2$, $b=2$, but $f(n)$ is not a polynomial. However, we have $f(n) \in \Theta(n \log n)$, $k=1$
- Therefore by the 4th condition of the Master Theorem we can say that

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n^{\log_2 2} \log^2 n) = \Theta(n \log^2 n)$$

Summary

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