# Functions

**Section 2.3 of Rosen**

Spring 2018

CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

# Outline

- Definitions & terminology
  - function, domain, co-domain, image, preimage (antecedent), range, image of a set, strictly increasing, strictly decreasing, monotonic
- Properties
  - One-to-one (injective)
  - Onto (surjective)
  - One-to-one correspondence (bijective)
  - Exercices (5)
- Inverse functions (examples)
- Operators
  - Composition, Equality
- Important functions
  - identity, absolute value, floor, ceiling, factorial

# Introduction

- You have already encountered function
  - $f(x,y) = x+y$
  - $f(x) = x$
  - $f(x) = \sin(x)$
- Here we will study functions defined on discrete domains and ranges
- We may not always be able to write function in a 'neat way' as above

# Definition: Function

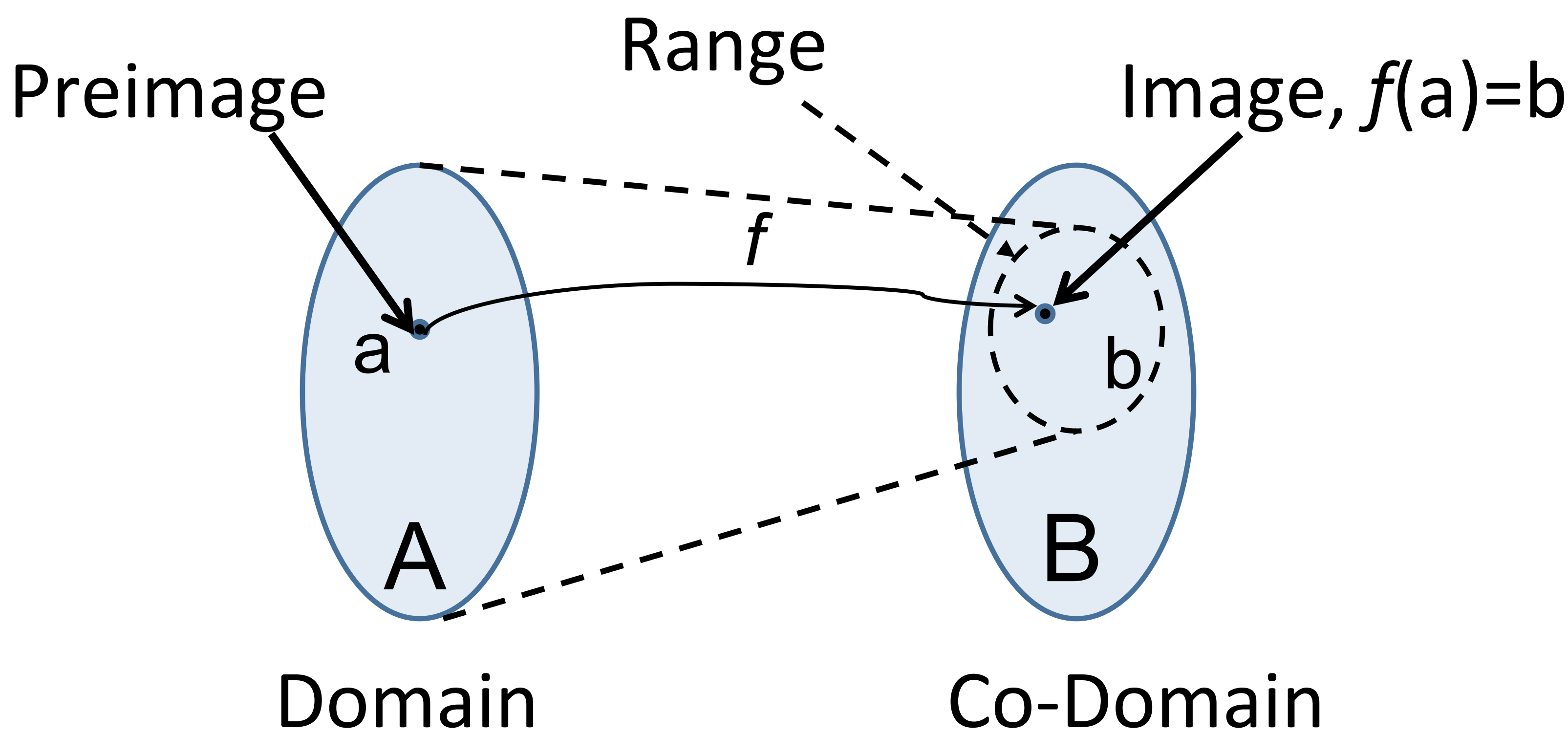
- **Definition**: A function f
  - from a set A to a set B
  - is an assignment of exactly one element of B to each element of A.
- We write $f$(a)=b if b is the unique element of B assigned by the function f to the element a $\in$ A.
- Notation: ***f*: A → B**

  which can be read as '*f* maps A to B'
- Note the subtlety
  - Each and every element of A has a single mapping
  - Each element of B may be mapped to by several elements in A or not at all

# Terminology

- Let $f$: A $\longrightarrow$ B and $f$(a)=b. Then we use the following terminology:
  - A is the <u>domain</u> of $f$, denoted dom($f$)
  - B is the <u>co-domain</u> of $f$
  - b is the <u>image</u> of a
  - a is the <u>preimage</u> (<u>antecedent</u>) of b
  - The <u>range</u> of $f$ is the set of all images of elements of A, denoted rng($f$)

# Function: Visualization




A function, $f$: A $\rightarrow$ B

# More Definitions (1)

- **Definition**: Let $f_1$ and $f_2$ be two functions from a set A to R. Then $f_1+f_2$ and $f_1f_2$ are also function from A to R defined by:
  - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
  - $f_1f_2(x)= f_1(x)f_2(x)$
- Example: Let $f_1(x)=x^4+2x^2+1$ and $f_2(x)=2-x^2$
  - $(f_1+f_2)(x) = x^4+2x^2+1+2-x^2 = x^4+x^2+3$
  - $f_1f_2(x) = (x^4+2x^2+1)(2-x^2)= -x^6+3x^2+2$

# More Definitions (2)

- **Definition**: Let $f$: A$\rightarrow$ B and S $\subseteq$A. The image of the set S is the subset of B that consists of all the images of the elements of S. We denote the image of S by $f$(S), so that

$$f(S)=\{ f(s) \mid \forall\ s \in S \}$$

- Note there that the image of S is a set and not an element.

# Image of a set: Example

- Let:
  - $A = \{a_1,a_2,a_3,a_4,a_5\}$
  - $B = \{b_1,b_2,b_3,b_4,b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
- Draw a diagram for *f*
- What is the:
  - Domain, co-domain, range of *f*?
  - Image of S, *f*(S)?

# More Definitions (3)

- **Definition**: A function *f* whose domain and codomain are subsets of the set of real numbers (*R*) is called
  - strictly increasing if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of *f*.
  - strictly decreasing if $f(x)>f(y)$ whenever $x<y$ and x and y are in the domain of *f*.
- A function that is increasing or decreasing is said to be monotonic

# Outline

- Definitions & terminology
- **Properties**
  - **One-to-one (injective)**
  - **Onto (surjective)**
  - **One-to-one correspondence (bijective)**
  - **Exercices (5)**
- Inverse functions (examples)
- Operators
- Important functions

# Definition: Injection

- **Definition**: A function $f$ is said to be one-to-one or injective (or an injection) if

$$\forall \text{ x and y in in the domain of } f,\ f(x)=f(y) \Rightarrow x=y$$

- Intuitively, an injection simply means that each element in the range has at most one preimage (antecedent)
- It is useful to think of the contrapositive of this definition

$$x \neq y \Rightarrow f(x) \neq f(y)$$

# Definition: Surjection

- **Definition**: A function $f$: $A \rightarrow B$ is called onto or surjective (or an surjection) if

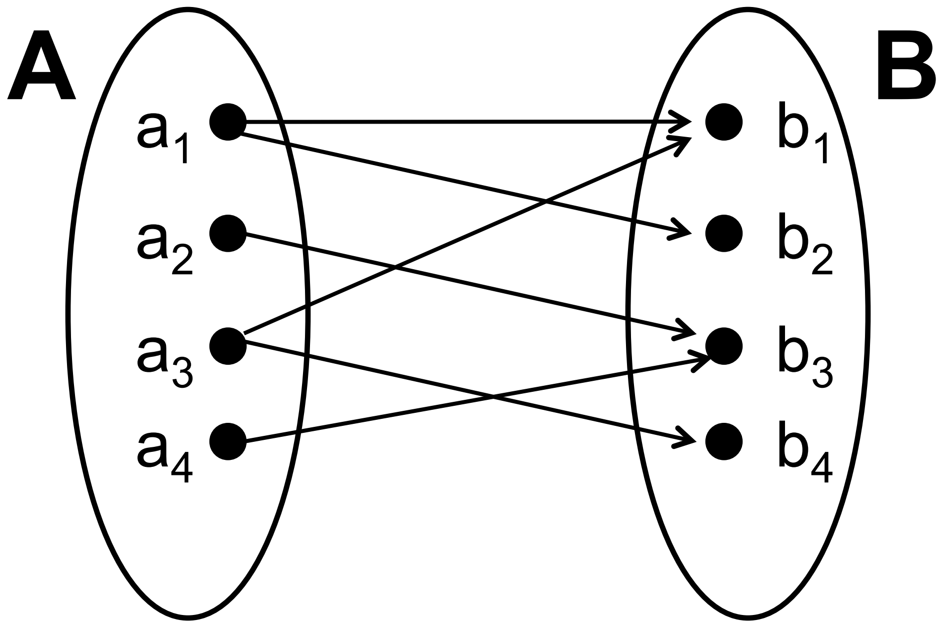
$$\forall\, b \in B,\ \exists\, a \in A \text{ with } f(a)=b$$

- Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)
- Thus, the range is the same as the codomain

# Definition: Bijection

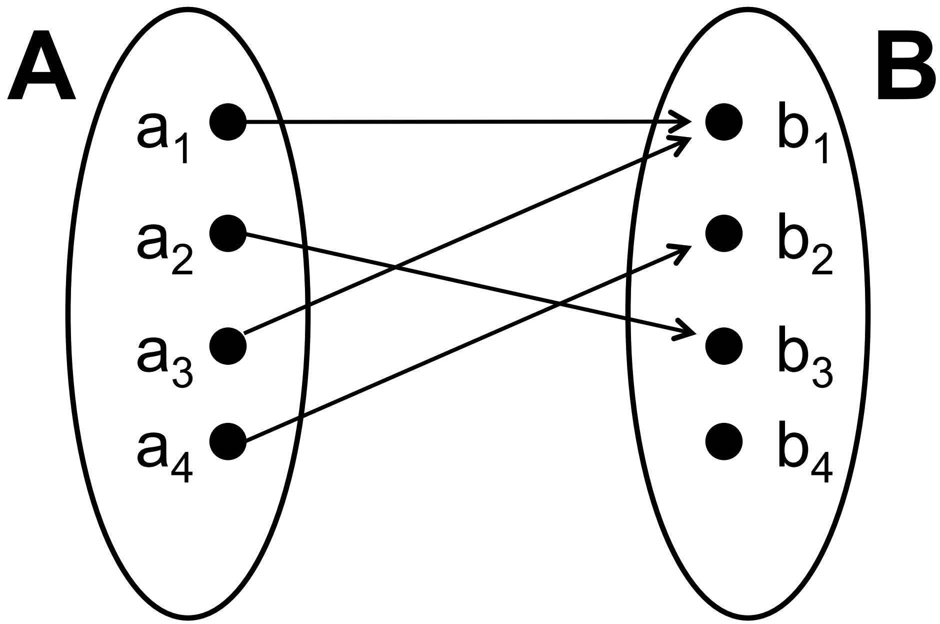
- **Definition**: A function *f* is a <u>one-to-one</u> correspondence (or a <u>bijection</u>), if it is both
  - one-to-one (injective) and
  - onto (surjective)
- One-to-one correspondences are important because they endow a function with an <u>inverse</u>.
- They also allow us to have a concept cardinality for infinite sets
- Let's look at a few examples to develop a feel for these definitions…

# Functions: Example 1



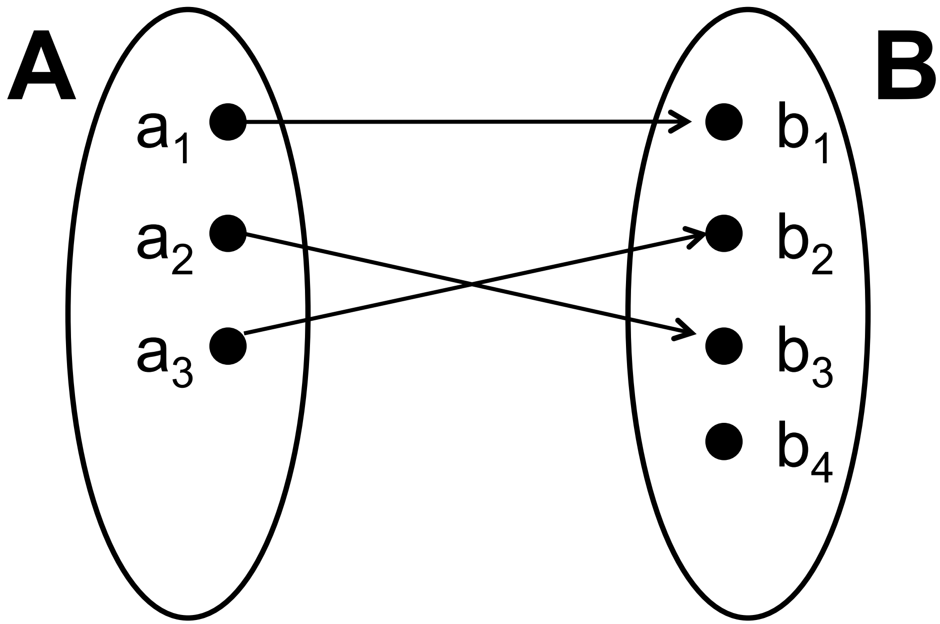

- Is this a function? Why?
- No, because each of $a_1$, $a_2$ has two images

# Functions: Example 2



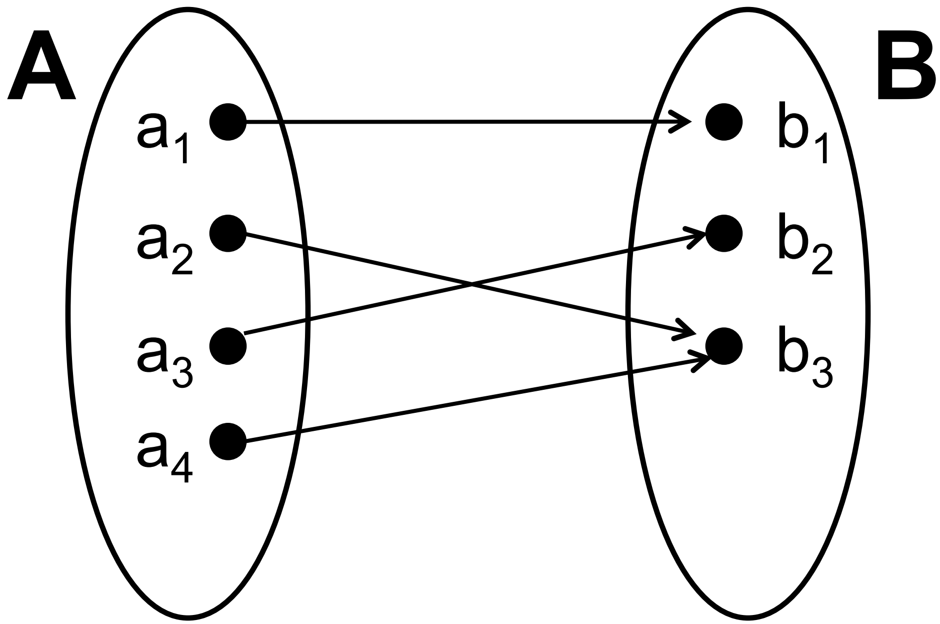

- Is this a function
  - One-to-one (injective)? Why? No, $b_1$ has 2 preimages
  - Onto (surjective)? Why? No, $b_4$ has no preimage

# Functions: Example 3



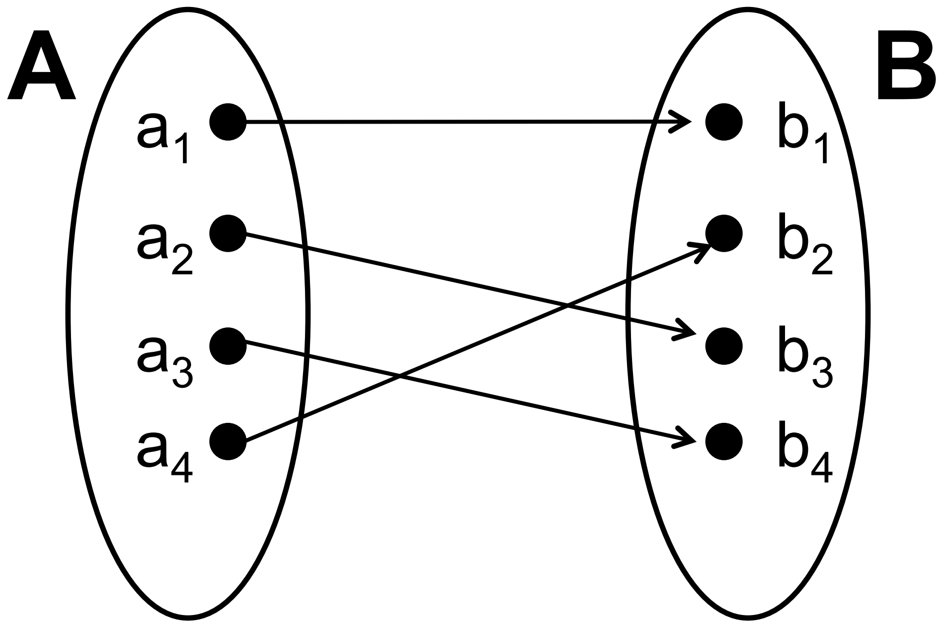

- Is this a function
  - One-to-one (injective)? Why? Yes, no $b_i$ has 2 preimages
  - Onto (surjective)? Why? No, $b_4$ has no preimage

# Functions: Example 4




- Is this a function
  - One-to-one (injective)? Why? No, $b_3$ has 2 preimages
  - Onto (surjective)? Why? Yes, every $b_i$ has a preimage

# Functions: Example 5




- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence

# Exercice 1

- Let $f$:$Z \rightarrow Z$ be defined by

$$f(x)=2x-3$$

- What is the domain, codomain, range of $f$?
- Is $f$ one-to-one (injective)?
- Is $f$ onto (surjective)?
- Clearly, dom($f$)=$Z$. To see what the range is, note that:

$$
\begin{aligned}
b \in \mathrm{rng}(f) &\Leftrightarrow b=2a-3, \text{ with } a \in Z \\
&\Leftrightarrow b=2(a-2)+1 \\
&\Leftrightarrow b \text{ is odd}
\end{aligned}
$$

# Exercise 1 (cont’d)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., rng($f$) $\neq Z$), we can conclude that $f$ is not onto (surjective)
- However, $f$ is one-to-one injective. Using simple algebra, we have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1\text{-}3 = 2x_2\text{-}3 \Rightarrow x_1 = x_2 \quad \text{QED}$$

# Exercise 2

- Let $f$ be as before

$$f(x)=2x-3$$

  but now we define $f: N \rightarrow N$
- What is the domain and range of $f$?
- Is $f$ onto (surjective)?
- Is $f$ one-to-one (injective)?

- By changing the domain and codomain of $f$, $f$ is not even a function anymore. Indeed, $f(1)=2\cdot1-3=-1\notin N$

# Exercice 3

- Let $f$:$Z \rightarrow Z$ be defined by

$$f(x) = x^2 - 5x + 5$$

- Is this function
  - One-to-one?
  - Onto?

# Exercice 3: Answer

- It is not one-to-one (injective)

$f(x_1)=f(x_2) \Rightarrow x_1^2-5x_1+5=x_2^2 - 5x_2+5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2$

$\Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2)$

$\Rightarrow (x_1 + x_2) = 5$

Many $x_1,x_2 \in Z$ satisfy this equality. There are thus an infinite number of solutions. In particular, $f(2)=f(3)=-1$

- It is also not onto (surjective).

The function is a parabola with a global minimum at (5/2,-5/4). Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/codomain?

# Exercice 4

- Let $f$:$Z \rightarrow Z$ be defined by

$$f(x) = 2x^2 + 7x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?
- Again, this is a parabola, it cannot be onto (where is the global minimum?)

# Exercice 4: Answer

- f(x) is one-to-one! Indeed:

$f(x_1)=f(x_2) \Rightarrow 2x_1^2+7x_1=2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1$

$\Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 + x_2) = -7$

$\Rightarrow (x_1 + x_2) = -7/2$

But $-7/2 \notin Z$. Therefore it must be the case that $x_1 = x_2$.

It follows that $f$ is a one-to-one function. QED

- f(x) is not surjective because f(x)=1 does not exist

$2x^2 +7x=1 \Rightarrow x(2x +7)=1$ the product of two integers is 1 if both integers are 1 or -1

$x=1 \Rightarrow (2x+7)=1 \Rightarrow 9 =1$, impossible

$x=-1 \Rightarrow -1(-2+7)=1 \Rightarrow -5=1$, impossible

# Exercise 5

- Let $f$:$Z \rightarrow Z$ be defined by

$$f(x) = 3x^3 - x$$

- Is this function
  - One-to-one (injective)?
  - Onto (surjective)?

# Exercice 5: *f* is one-to-one

- To check if *f* is one-to-one, again we suppose that for $x_1, x_2 \in Z$ we have $f(x_1)=f(x_2)$

$f(x_1)=f(x_2) \Rightarrow 3x_1^3 - x_1 = 3x_2^3 - x_2$

$\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$

$\Rightarrow 3\,(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$

$\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = 1/3$

which is impossible because $x_1, x_2 \in Z$

thus, *f* is one-to-one

# Exercice 5: $f$ is <u>not</u> onto

- Consider the counter example $f(a)=1$
- If this were true, we would have

  $3a^3 - a=1 \Rightarrow a(3a^2 - 1)=1$ where a and $(3a^2 - 1) \in \mathbb{Z}$
- The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1
- Neither 1 nor -1 satisfy the above equality
  - Thus, we have identified $1 \in \mathbb{Z}$ that does not have an antecedent and $f$ is not onto (surjective)

# Outline

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- Properties
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  - Exercices (5)
- **Inverse functions (examples)**
- **Operators**
  - **Composition, Equality**
- Important functions
  - identity, absolute value, floor, ceiling, factorial

# Inverse Functions (1)

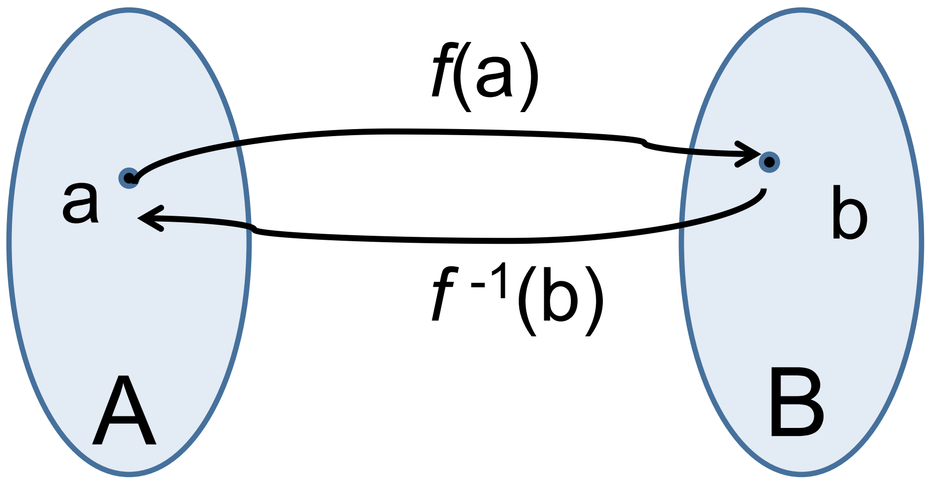
- **Definition**: Let $f$: $A \rightarrow B$ be a bijection. The <u>inverse</u> function of $f$ is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a)=b$
- The inverse function is denote $f^{-1}$.
- When $f$ is a bijection, its inverse exists and

$$f(a)=b \Leftrightarrow f^{-1}(b)=a$$

# Inverse Functions (2)

- Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is <u>invertible</u>
- Why must a function be bijective to have an inverse?
  - Consider the case where f is not one-to-one (not injective). This means that some element $b \in B$ has more than one antecedent in A, say $a_1$ and $a_2$. How can we define an inverse? Does $f^{-1}(b)=a_1$ or $a_2$?
  - Consider the case where f is not onto (not surjective). This means that there is some element $b \in B$ that does not have any preimage $a \in A$. What is then $f^{-1}(b)$?

# Inverse Functions: Representation




Domain

Co-Domain

A function and its inverse

# Inverse Functions: Example 1

- Let $f$:$R \rightarrow R$ be defined by

$$f(x) = 2x - 3$$

- What is $f^{-1}$?
  1. We must verify that $f$ is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.
  2. To find the inverse, we use the substitution
     - Let $f^{-1}(y)=x$
     - And y=2x-3, which we solve for x. Clearly, x= (y+3)/2
     - So, $f^{-1}(y)= (y+3)/2$

# Inverse Functions: Example 2

- Let $f(x)=x^2$. What is $f^{-1}$?
- No domain/codomain has been specified.
- Say $f: R \rightarrow R$
  - Is $f$ a bijection? Does its inverse exist?
  - Answer: No
- Say we specify that $f: A \rightarrow B$ where

$$A=\{x \in R \mid x \leq 0\} \text{ and } B=\{y \in R \mid y \geq 0\}$$

  - Is $f$ a bijection? Does its inverse exist?
  - Answer: Yes, the function <u>becomes</u> a bijection and thus, has an inverse

# Inverse Functions: Example 2 (cont')

- To find the inverse, we let
  - $f^{-1}(y)=x$
  - $y=x^2$, which we solve for x
- Solving for x, we get $x=\pm\sqrt{y}$, but which one is it?
- Since dom($f$) is all nonpositive and rng($f$) is nonnegative, thus x must be nonpositive and

$$f^{-1}(y)= -\sqrt{y}$$

- From this, we see that <u>the domains/codomains are just as important to a function as the definition of the function</u> itself

# Inverse Functions: Example 3

- Let $f(x)=2^x$
  - What should the domain/codomain be for this function to be a bijection?
  - What is the inverse?
- The function should be $f: R \rightarrow R^+$
- Let $f^{-1}(y)=x$ and $y=2^x$, solving for x we get $x=\log_2(y)$. Thus, $f^{-1}(y)=\log_2(y)$
- What happens when we include 0 in the codomain?
- What happens when restrict either sets to $Z$?

# Function Composition (1)

- The value of functions can be used as the input to other functions
- **Definition**: Let $g$:A$\longrightarrow$B and $f$:B $\longrightarrow$C. The <u>composition</u> of the functions f and g is

$$(f \circ g)\,(x)=f(g(x))$$

- $f \circ g$ is read as '*f* circle *g*', or '*f* composed with *g*', '*f* following *g*', or just '*f* of *g*'
- In LaTeX: $\circ$

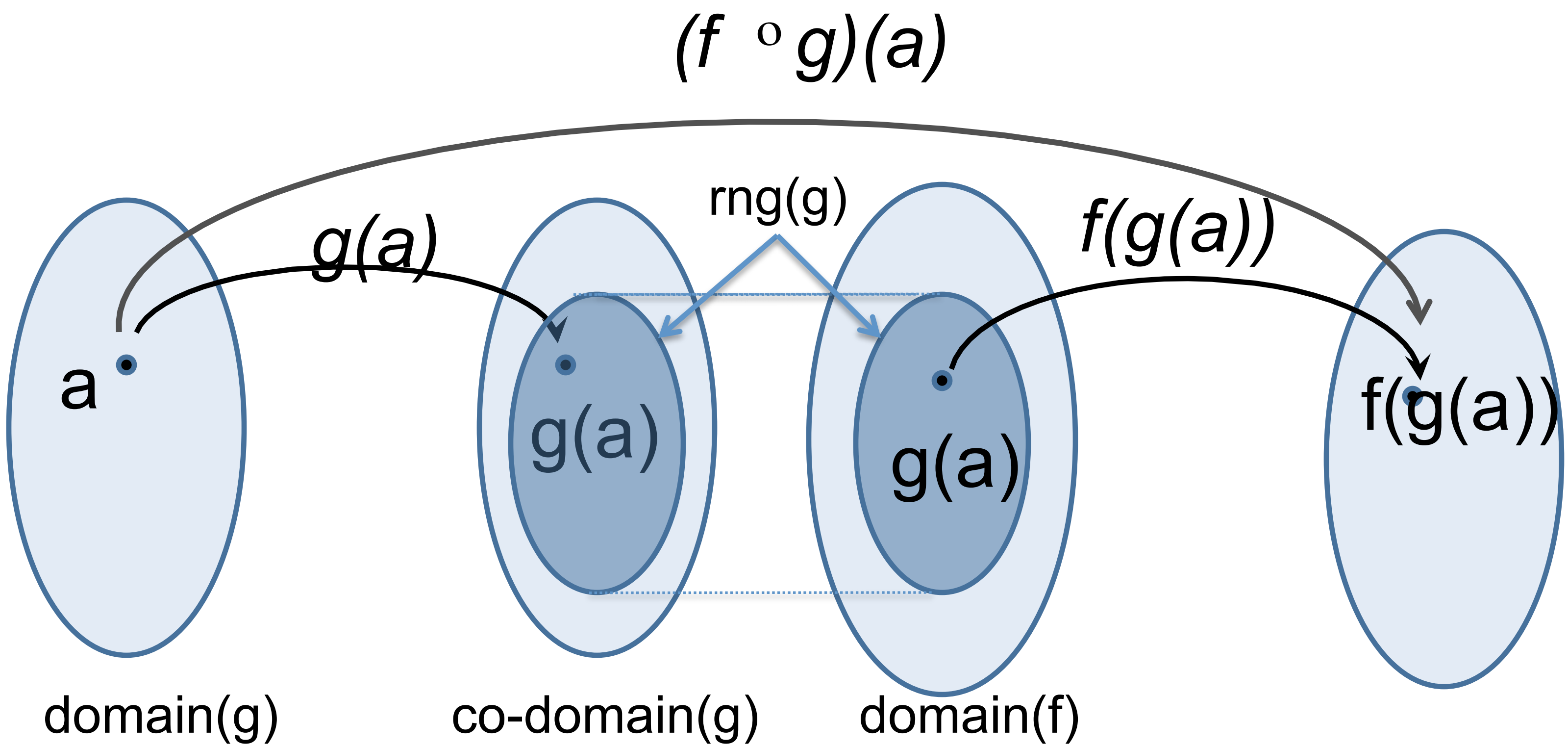
# Function Composition (2)

- Because $(f \circ g)(x)=f(g(x))$, the composition $f \circ g$ cannot be defined unless the range of $g$ is a subset of the domain of $f$

$$f \circ g \text{ is defined } \Leftrightarrow \text{rng}(g) \subseteq \text{dom}(f)$$

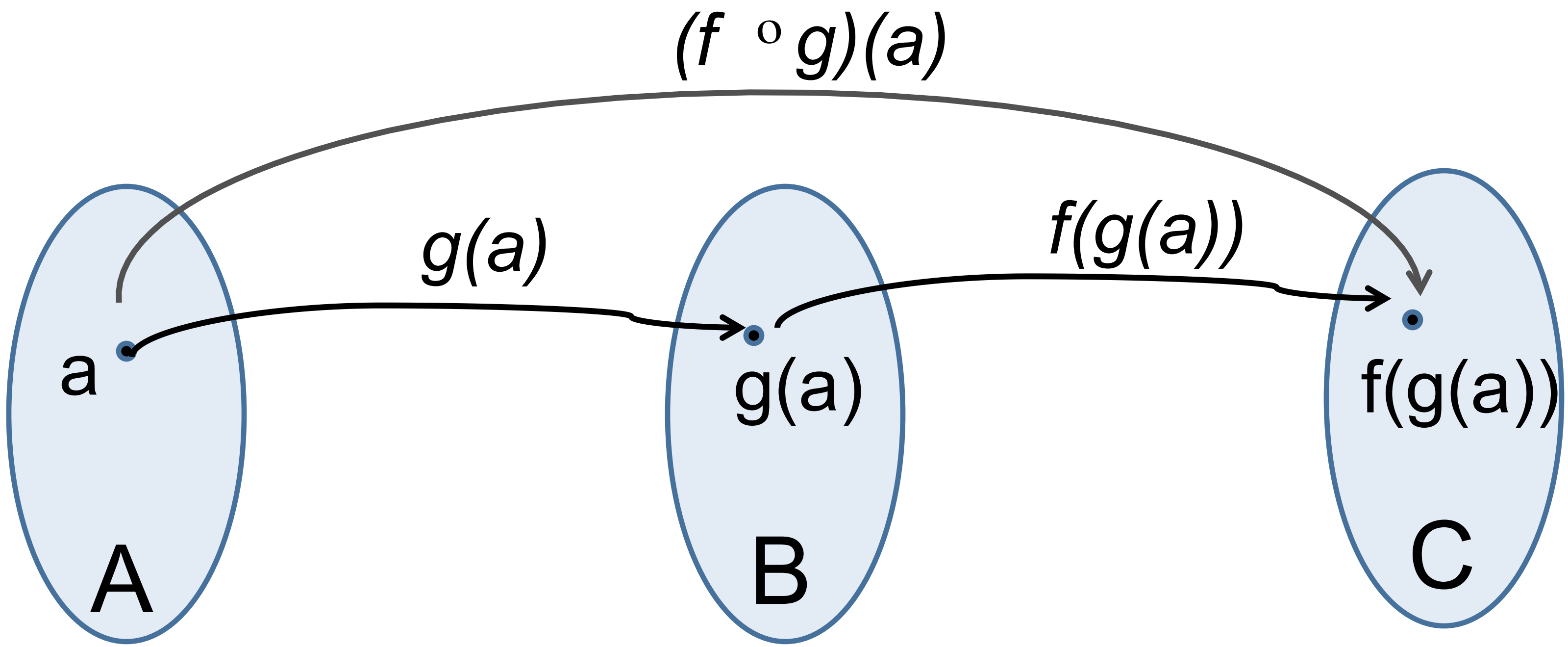
- The <u>order</u> in which you apply a function matters: you go from the inner most to the outer most
- It follows that $f \circ g$ is in general <u>not</u> the same as $g \circ f$

# Composition: Graphical Representation




The composition of two functions

# Composition: Graphical Representation




The composition of two functions

# Composition: Example 1

- Let $f$, $g$ be two functions on $R \rightarrow R$ defined by

$$f(x) = 2x - 3$$

$$g(x) = x^2 + 1$$

- What are $f \circ g$ and $g \circ f$?
- We note that
  - $f$ is bijective, thus dom($f$)=rng($f$)= codomain($f$)= $R$
  - For $g$, dom($g$)= $R$ but rng($g$)=$\{x \in R \mid x \geq 1\} \subseteq R^+$
  - Since rng($g$)=$\{x \in R \mid x \geq 1\} \subseteq R^+ \subseteq$ dom($f$) =$R$, $f \circ g$ is defined
  - Since rng($f$)= $R \subseteq$ dom($g$) =$R$ , $g \circ f$ is defined

# Composition: Example 1 (cont')

- Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$
- $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$

  $= 2x^2 - 1$
- $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 + 1$

  $= 4x^2 - 12x + 10$

# Function Equality

- Although it is intuitive, we formally define what it means for two functions to be equal
- **Lemma**: Two functions $f$ and $g$ are equal if and only
  - dom($f$) = dom($g$)
  - $\forall$ a $\in$dom($f$) ($f$(a) = $g$(a))

# Associativity

- The composition of function is not commutative ($f \circ g \neq g \circ f$), it is associative
- **Lemma**: The composition of functions is an associative operation, that is

$$(f \circ g) \circ h = f \circ (g \circ h)$$

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- **Important functions**
  - **identity, absolute value, floor, ceiling, factorial**

# Important Functions: Identity

- **Definition**: The identity function on a set A is the function

  $$\iota: A \to A$$

  $\iota$

  defined by $\iota(a)=a$ for all $a \in A$.
- One can view the identity function as a composition of a function and its inverse:

  $$\iota(a) = (f \circ f^{-1})(a) = (f^{-1} \circ f)(a)$$
- Moreover, the composition of any function $f$ with the identity function is itself $f$:

  $$(f \circ \iota)(a) = (\iota \circ f)(a) = f(a)$$

# Inverses and Identity

- The identity function, along with the composition operation, gives us another characterization of <u>inverses</u> when a function has an inverse
- **Theorem**: The functions $f$: A→B and $g$: B→A are <u>inverses</u> if and only if

$$(g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B$$

where the $\iota_A$ and $\iota_B$ are the identity functions on sets A and B. That is,

$$\forall a \in A, b \in B \; (\, (g(f(a)) = a) \wedge (f(g(b)) = b) \,)$$

# Important Functions: Absolute Value

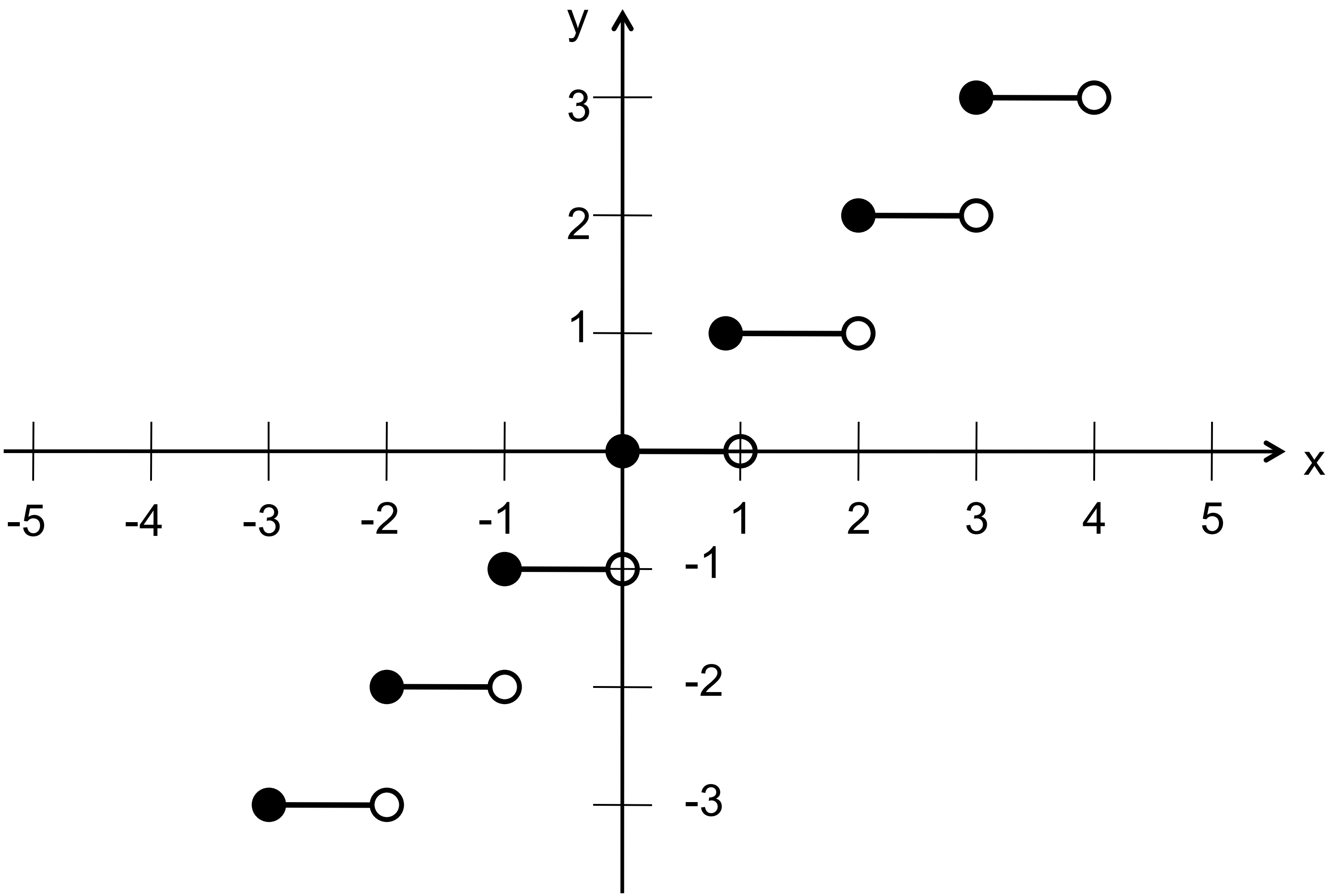
- **Definition**: The absolute value function, denoted $|x|$, f $f$:$R \rightarrow \{y \in R \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

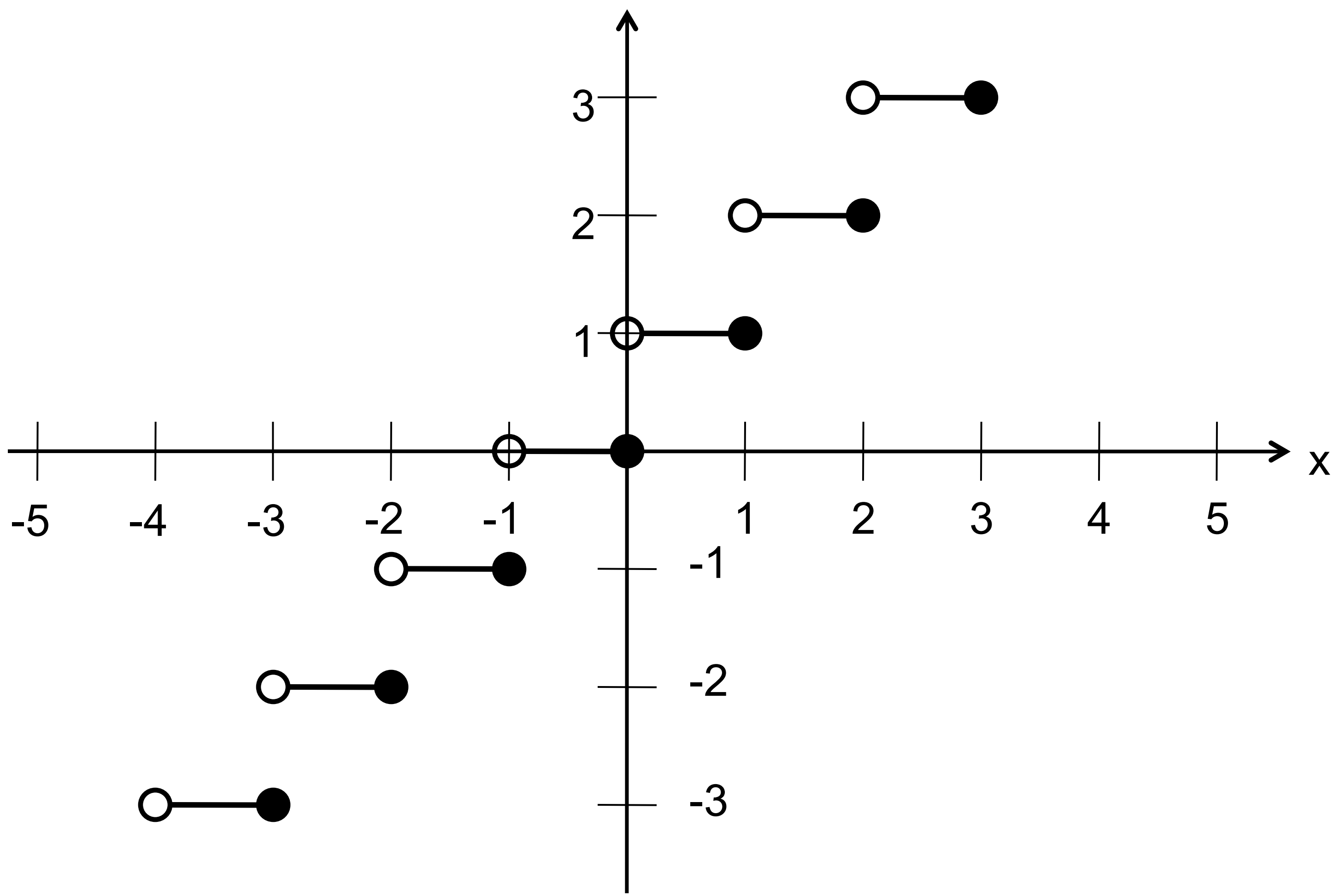
# Important Functions: Floor & Ceiling

- **Definitions**:
  - The floor function, denoted $\lfloor x \rfloor$, is a function $R \rightarrow Z$. Its values is the largest integer that is less than or equal to x
  - The ceiling function, denoted $\lceil x \rceil$, is a function $R \rightarrow Z$. Its values is the smallest integer that is greater than or equal to x
- In LaTex: $\lceil$, $\rceil$, $\rfloor$, $\lfloor$

# Important Functions: Floor



# Important Functions: Ceiling



# Important Function: Factorial

- The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of n objects
- **Definition**: The <u>factorial</u> function, denoted n!, is a function $N \rightarrow N^+$. Its value is the <u>product</u> of the n positive integers

$$n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$$

# Factorial Function & Stirling's Approximation

- The factorial function is defined on a discrete domain
- In many applications, it is useful a continuous version of the function (say if we want to differentiate it)
- To this end, we have the Stirling's formula

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

# Summary

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