Asymptotics

Section 3.2 of Rosen

Spring 2018

CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

Outline

- Introduction
- Asymptotic
  - Definitions (Big O, Omega, Theta), properties
- Proof techniques
  - 3 examples, trick for polynomials of degree 2,
  - Limit method (l'Hôpital Rule), 2 examples
- Limit Properties
- Complexity of algorithms
- Conclusions

Introduction (1)

- We are interested <u>only</u> in the <u>Order of Growth</u> of an algorithm's complexity
- How well does the algorithm perform as the size of the input grows: $n \rightarrow \infty$
- We have seen how to mathematically evaluate the cost functions of algorithms with respect to
  - their input size n and
  - their elementary operations
- However, it suffices to simply measure a cost function's <u>asymptotic behavior</u>

Introduction (2): Magnitude Graph

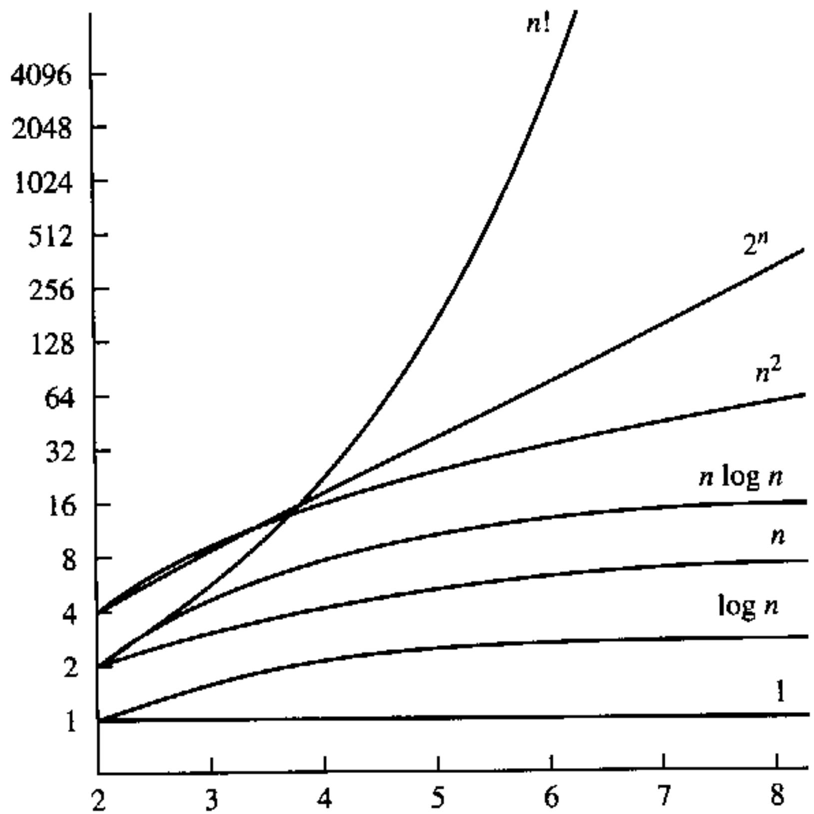

FIGURE 3 A Display of the Growth of Functions Commonly Used in Big-*O* Estimates.

Introduction (3)

- In practice, specific hardware, implementation, languages, etc. greatly affect how the algorithm behave
- Our goal is to study and analyze the behavior of algorithms *in and of themselves*, independently of such factors
- For example
  - An algorithm that executes its elementary operation $10n$ times is better than one that executes it $0.005n^2$ times
  - Also, algorithms that have running time n^2 and $2000n^2$ are considered asymptotically equivalent

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- Introduction
- **Asymptotic**
  - **Definitions (Big-O, Omega, Theta), properties**
- Proof techniques
- Limit Properties
- Efficiency classes
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Big-O Definition

- **Definition**: Let f and g be two functions $f,g:\mathbb{N} \to \mathbb{R}^+$. We say that

$$f(n) \in O(g(n))$$

(read: f is Big-O of g) if there exists a constant $c \in \mathbb{R}^+$ and an $n_o \in \mathbb{N}$ such that for every integer $n \geq n_0$ we have

$$f(n) \leq cg(n)$$

- Big-O is actually Omicron, but it suffices to write "O"

  Intuition: f is asymptotically less than or equal to g

- Big-O gives an asymptotic <u>upper bound</u> \mathcal{O}

Big-Omega Definition

- **Definition**: Let *f* and *g* be two functions $f,g: N \rightarrow R^+$. We say that

  $$f(n) \in \Omega(g(n))$$

  (read: *f* is Big-Omega of *g*) if there exists a constant $c \in R^+$ and an $n_o \in N$ such that for every integer $n \geq n_0$ we have

  $$f(n) \geq cg(n)$$

- Intuition: *f* is asymptotically greater than or equal to *g*
- Big-Omega gives an asymptotic lower bound \Omega()

Big-Theta Definition

- **Definition**: Let f and g be two functions f,g: $N \rightarrow R^+$. We say that

  $$f(n) \in \Theta(g(n))$$

  (read: f is Big-Omega of g) if there exists a constant c_1, $c_2 \in R^+$ and an $n_o \in N$ such that for every integer $n \geq n_0$ we have

  $$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

- Intuition: f is asymptotically equal to g
- f is bounded above and below by g
- Big-Theta gives an asymptotic <u>equivalence</u> \Theta ()

Asymptotic Properties (1)

- **Theorem**: For $f_1(n) \in O(g_1(n))$ and $f_2(n) \in O(g_2(n))$, we have
$$f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$$
- This property implies that we can ignore lower order terms. In particular, for any polynomial with degree k such as $p(n)=an^k + bn^{k-1} + cn^{k-2} + \ldots$,
$$p(n) \in O(n^k)$$

More accurately, $p(n) \in \Theta(n^k)$

- In addition, this theorem gives us a justification for ignoring constant coefficients. That is for any function $f(n)$ and a positive constant c
$$cf(n) \in \Theta(f(n))$$

Asymptotic Properties (2)

- Some obvious properties also follow from the definitions
- **Corollary**: For positive functions $f(\mathrm{n})$ and $g(\mathrm{n})$ the following hold:
  - $f(\mathrm{n}) \in \Theta(g(\mathrm{n})) \Leftrightarrow f(\mathrm{n}) \in O(g(\mathrm{n})) \wedge f(\mathrm{n}) \in \Omega(g(\mathrm{n}))$
  - $f(\mathrm{n}) \in O(g(\mathrm{n})) \Leftrightarrow g(\mathrm{n}) \in \Omega(f(\mathrm{n}))$

  The proof is obvious and left as an exercise

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Asymptotic Proof Techniques

- Proving an asymptotic relationship between two given function *f*(n) and *g*(n) can be done intuitively for most of the functions you will encounter; all polynomials for example
- However, this *<u>does not suffice</u>* as a formal proof
- To prove a relationship of the form $f(n) \in \Delta(g(n))$, where Δ is O, Ω, or Θ, can be done using the definitions, that is
  - **Find a value for c (or c_1 and c_2)**
  - **Find a value for n_0**

  (But the above is not the only way.)

Asymptotic Proof Techniques: Example A

Example: Let $f(n)=21n^2+n$ and $g(n)=n^3$

- Our intuition should tell us that $f(n) \in O(g(n))$
- Simply using the definition confirms this:

$$21n^2+n \leq cn^3$$

holds for say $c=3$ and for all $n \geq n_0=8$

- So we found a pair $c=3$ and $n_0=8$ that satisfy the conditions required by the definition **QED**
- In fact, an infinite number of pairs can satisfy this equation

Asymptotic Proof Techniques: Example B (1)

- **Example**: Let $f(n)=n^2+n$ and $g(n)=n^3$. Find a tight bound of the form

$$f(n) \in \Delta(g(n))$$

- Our intuition tells us that $f(n) \in O(g(n))$
- Let's prove it formally

Example B: Proof

- If $n \geq 1$ it is clear that
  1. $n \leq n^3$ and
  2. $n^2 \leq n^3$
- Therefore, we have, as 1. and 2.:

$$n^2+n \leq n^3 + n^3 = 2n^3$$

- Thus, for $n_0=1$ and $c=2$, by the definition of Big-O we have that $f(n)=n^2+n \in O(g(n^3))$

Asymptotic Proof Techniques: Example C (1)

- **Example**: Let $f(n)=n^3+4n^2$ and $g(n)=n^2$. Find a tight bound of the form

$$f(n) \in \Delta(g(n))$$

- Here, 0ur intuition tells us that $f(n)\in\Omega(g(n))$
- Let's prove it formally

Example C: Proof

- For $n \geq 1$, we have $n^2 \leq n^3$
- For $n \geq 0$, we have $n^3 \leq n^3 + 4n^2$
- Thus $n \geq 1$, we have $n^2 \leq n^3 \leq n^3 + 4n^2$
- Thus, by the definition of Big-Ω , for $n_0=1$ and $c=1$ we have that $f(n)=n^3+4n^2 \in \Omega(g(n^2))$

Asymptotic Proof Techniques: Trick for polynomials of degree 2

- If you have a polynomial of degree 2 such as

  $$an^2+bn+c$$

  you can prove that it is $\Theta(n^2)$ using the following values

  1. $c_1=a/4$
  2. $c_2=7a/4$
  3. $n_0= 2 \max(|b|/a, \text{sqrt}(|c|)/a)$

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Limit Method: Motivation

- Now try this one:

  $f(n) = n^{50} + 12n^3\log^4 n - 1243n^{12}$

  $+ 245n^6\log n + 12\log^3 n - \log n$

  $g(n) = 12\, n^{50} + 24\, \log^{14} n^{43} - \log n / n^5 + 12$
- Using the formal definitions can be very tedious especially one has very complex functions
- It is much better to use the Limit Method, which uses concepts from Calculus

Limit Method: The Process

- Say we have functions f(n) and g(n). We set up a limit quotient between f and g as follows

$$\lim_{n\to\infty} f(n)/g(n) = \begin{cases} 0 & \text{Then } f(n)\in O(g(n)) \\ c>0 & \text{Then } f(n)\in \Theta(g(n)) \\ \infty & \text{Then } f(n)\in \Omega(g(n)) \end{cases}$$

- The above can be proven using calculus, but for our purposes, the limit method is sufficient for showing asymptotic inclusions
- Always try to look for algebraic simplifications first
- If f and g both diverge or converge on zero or infinity, then you need to apply the l'Hôpital Rule

(Guillaume de) L'Hôpital Rule

- Theorem (L'Hôpital Rule):
  - Let f and g be two functions,
  - if the limit between the quotient f(n)/g(n) exists,
  - Then, it is equal to the limit of the derivative of the numerator and the denominator

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} f'(n)/g'(n)$$

Useful Identities & Derivatives

- Some useful derivatives that you should memorize
  - $(n^k)' = k\, n^{k-1}$
  - $(\log_b (n))' = 1/(n \ln(b))$
  - $(f_1(n)f_2(n))' = f_1'(n)f_2(n)+f_1(n)f_2'(n)$ *(product rule)*
  - $(\log_b(f(n))' = f'(n)/(f(n).\ln b)$
  - $(c^n)' = \ln(c)c^n$ ← *careful*!
- Log identities
  - Change of base formula: $\log_b(n)= \log_c(n)/\log_c(b)$
  - $\log(n^k) = k \log(n)$
  - $\log(ab) = \log(a) + \log(b)$

L'Hôpital Rule: Justification (1)

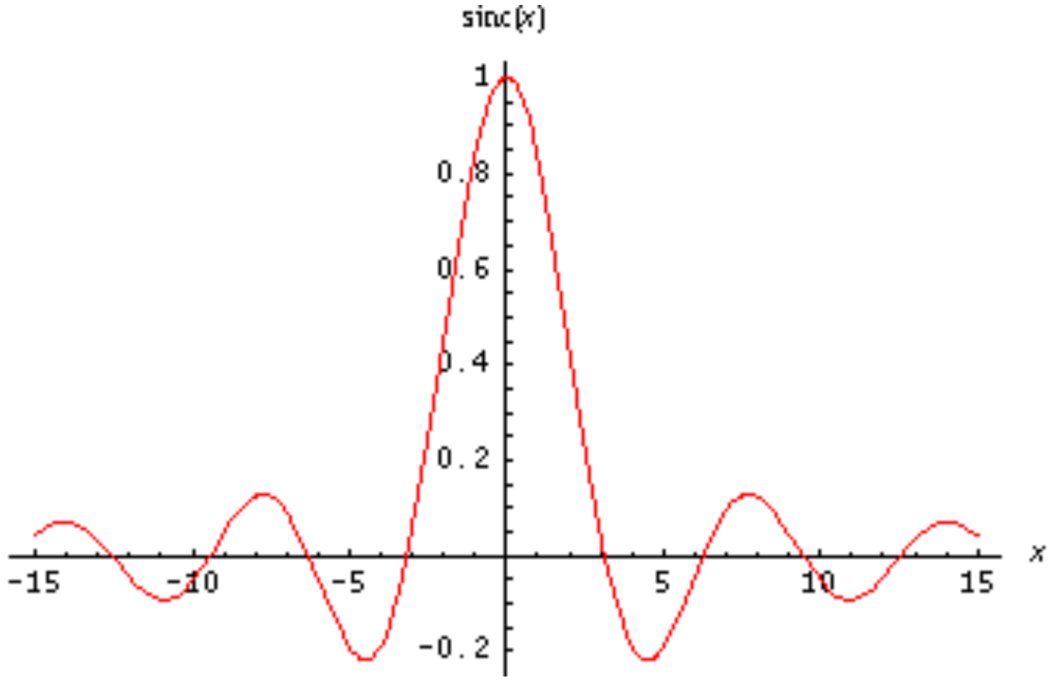
- Why do we have to use L'Hôpital's Rule?
- Consider the following function

$$f(x)= (\sin x)/x$$

- Clearly sin 0=0. So you may say that when $x \rightarrow 0$, $f(x) \rightarrow 0$
- However, the denominator is also $\rightarrow 0$, so you may say that $f(x) \rightarrow \infty$
- Both are wrong

L'Hôpital Rule: Justification (2)

- Observe the graph of f(x)= (sin x)/x = sinc x



L'Hôpital Rule: Justification (3)

- Clearly, though f(x) is undefined at x=0, the limit still exists
- Applying the L'Hôpital Rule gives us the correct answer

$$\lim_{x\to 0} ((\sin x)/x) = \lim_{x\to 0} (\sin x)' / x' = \cos x/1 = 1$$

Limit Method: Example 1

- Example: Let $f(n) = 2^n$, $g(n) = 3^n$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$
- What is your intuition in this case? Which function grows quicker?

Limit Method: Example 1—Proof A

- Proof using limits
- We set up our limit:

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} 2^n/3^n$$

- Using L' Hôpital Rule <u>gets you no where</u>

$$\lim_{n\to\infty} 2^n/3^n = \lim_{n\to\infty} (2^n)'/(3^n)' = \lim_{n\to\infty} (\ln 2)(2^n)/(\ln 3)(3^n)$$

- Both the numerator and denominator still diverge. We'll have to use an algebraic simplification

Limit Method: Example 1—Proof B

- Using algebra

$$\lim_{n\to\infty} 2^n/3^n = \lim_{n\to\infty} (2/3)^n$$

- Now we use the following Theorem w/o proof

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

- Therefore we conclude that the $\lim_{n\to\infty} (2/3)^n$ converges to zero thus $2^n \in O(3^n)$

Limit Method: Example 2 (1)

- Example: Let $f(n) = \log_2 n$, $g(n) = \log_3 n^2$. Determine a tight inclusion of the form

$$f(n) \in \Delta(g(n))$$

- What is your intuition in this case?

Limit Method: Example 2 (2)

- We prove using limits
- We set up out limit

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} \log_2 n/\log_3 n^2$$
$$= \lim_{n\to\infty} \log_2 n/(2\log_3 n)$$

- Here we use the change of base formula for logarithms: $\log_x n = \log_y n/\log_y x$
- Thus: $\log_3 n = \log_2 n / \log_2 3$

Limit Method: Example 2 (3)

- Computing our limit:

$$\lim_{n\to\infty} \log_2 n/(2\log_3 n) = \lim_{n\to\infty} \log_2 n \log_2 3 /(2\log_2 n)$$
$$= \lim_{n\to\infty} (\log_2 3)/2$$
$$= (\log_2 3) /2$$
$$\approx 0.7924, \text{ which is a positive constant}$$

- So we conclude that $f(n) \in \Theta(g(n))$

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Limit Properties

- A useful property of limits is that the composition of functions is preserved
- **Lemma**: For the composition $\circ$ of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

$$\lim_{n\to\infty} f_1(n) \circ \lim_{n\to\infty} f_2(n) = \lim_{n\to\infty} (f_1(n) \circ f_2(n))$$

Complexity of Algorithms—Table 1, page 226

• Constant	$O(1)$
• Logarithmic	$O(\log (n))$
• Linear	$O(n)$
• Polylogarithmic	$O(\log^k (n))$
• Quadratic	$O(n^2)$
• Cubic	$O(n^3)$
• Polynominal	$O(n^k)$ for any $k>0$
• Exponential	$O(k^n)$, where $k>1$
• Factorial	$O(n!)$

Conclusions

- Evaluating asymptotics is easy, but remember:
  - **Always** look for algebraic simplifications
  - You must **always** give a rigorous proof
  - Using the limit method is (almost) always the best
  - Use L'Hôpital Rule if need be
  - Give as simple **and tight** expressions as possible

Summary

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