Recitation 8

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- First, a few definitions:
  1. **Reflexive**: $(a, a) \in R$ for all $a \in A$
  2. **Symmetric**: $\forall a, b \in A$, $(a, b) \in R \rightarrow (b, a) \in R$
  3. **Antisymmetric**:$a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
  4. **Transitive**: $a, b, c \in A$, $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$
  5. **Irreflexive**: $\forall a \in A$, $(a, a) \notin R$
  6. **Asymmetric**: $(a, b) \in R$ then $(b, a) \notin R$
  7. **Equivalence Relation**: A relation that is *reflexive*, *symmetric*, and *transitive*.
  8. **Partial Ordering**: A relation **R** on a set **S** that is *reflexive, antisymmetric, and transitive*

- Rosen 9.4.25(c). Use Algorithm 1 (Given on page 603) to compute the transitive closure of the relation $\{(1, 2), (1, 3), (1, 4), (2, 3), (2.4), (3.4)\}$ on the set $\{1, 2, 3, 4\}$.
  - We note the matrix of a relation R^x resulting from the composing the relation R with itself x times: M_{R^x}, alternatively: $M_R^{[x]}$.
  - We note the relations composition operator $\circ$ and the matrix product operator $\cdot$, alternatively, $\odot$.

$$M_R = M_{R^1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = M_{R^1 \circ R^1} = M_{R^1} \cdot M_{R^1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = M_{R \circ R^2} = M_{R^2} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^4} = M_{R \circ R^3} = M_{R^3} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} M_{R^*} &= M_{R^1} \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So it was already transitive.

- Rosen 9.4.27(a)

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

So the transitive closure contains all 16 pairs. Is this transitive? **Yes**

- 9.5.3 a) Is $\{(f, g) \mid f(1) = g(0) \text{ and } f(0) = g(1)\}$ an equivalence relation on the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$?
  - So this relation, contains (f, g), if $f(1) = g(0)$ and $f(0) = g(1)$.
  - So is this relation Reflexive? No, $f(0) = f(0)$ and $f(1) = f(1)$, this is only true when $f(0) = f(1)$. But there are other cases, so this is not reflexive.

- Is this relation Symmetric?Yes, if $f(1) = g(0)$ and $f(0) = g(1)$, then $g(1) = f(0)$and $g(0) = f(1)$. So: if (f, g) then (g, f)
- Is this relation Transitive? Suppose we have (f, g) and (g, c). Then $f(1) = g(0)$ and $f(0) = g(1)$ and $g(1) = c(0)$ and $g(0) = c(1)$. Therefore $f(1) = c(1)$ and $f(0) = c(0)$ so it is not neccessarily true that $f(1) = c(0)$ and $f(0) = c(1)$. So no this is not transitive..

- 9.5.3 b) How about $\{(f, g) \mid f(1) = g(1) \text{ or } f(0) = g(0)\}$
  - Is it Reflexive? Yes if $f(1) = g(1)$, then $g(1) = f(1)$, same for $f(0)$ and $g(0)$
  - Is it Symmetric? Yes, again similar to last time.
  - Is it Transitive? No, suppose $f(1) = g(1)$ and $g(0) = c(0)$, but $f(0) \neq c(0)$ and $f(1) \neq c(1)$. Then we have (f, g) and (g, c), but we don't have (f, c).
- Equivalence Relations: 51 - Show that the partition of the set of bit strings of lengh 16 formed by equivalence classes of bit strings of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last 4 bits.
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- Partial orderings, 9.6.1 a){(0,0),(1,1),(2,2),(3,3)}
  - Yes, it is reflexive, antisymmetric, and transitive
- Partial orderings, 9.6.3 b) Let S be the set of all people in the world, and let aRb be the relation a is not taller than b. is this a partial ordering?
  - Reflexive: clearly a cannot be taller than a, so (a, a).
  - Antisymmetric: Suppose we have (a, b) is it possible to have (b, a) if $b \neq a$? Yes, suppose a and b have the same height, but are different people.
  - Transitive: Yes, if a is not taller than b and b is not taller than c then a is not taller than c.
  - So no, it is not antisymmetric, therefore it is not a partial ordering.