

Recitation 6

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- Problem 2.2:31: Show that for A and B subsets of some universal set U ,

$$A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$$

$$\begin{array}{ll}
 A \subseteq B \Leftrightarrow & \\
 \forall x, \quad x \in A \rightarrow x \in B & \text{Definition of set inclusion} \\
 \Leftrightarrow x \notin A \vee x \in B \Leftrightarrow & \text{Implication rule} \\
 x \in B \vee x \notin A \Leftrightarrow & \text{Commutativity} \\
 x \notin B \rightarrow x \notin A \Leftrightarrow & \text{Implication rule} \\
 x \in \bar{B} \rightarrow x \in \bar{A} \Leftrightarrow & \text{Definition of set complement} \\
 \bar{B} \subseteq \bar{A} & \text{by definition of a set inclusion} \\
 & \text{QED}
 \end{array}$$

- 2.2.37 c: Show that if A is a subset of universal set U

$$A \oplus U = \bar{A}$$

$$\begin{array}{ll}
 \forall x, \quad x \in A \oplus U \Leftrightarrow & \\
 ((x \in A) \vee (x \in U)) \wedge \neg((x \in A) \wedge (x \in U)) \Leftrightarrow & \text{definition of symmetric} \\
 & \text{difference } \oplus \text{ on page 137} \\
 ((x \in A \cup U)) \wedge \neg((x \in A \cap U)) \Leftrightarrow & \text{Definition set union, intersection} \\
 (x \in U) \wedge \neg(x \in A) \Leftrightarrow & \text{Domination, identity laws} \\
 \neg(x \in A) \wedge (x \in U) \Leftrightarrow & \text{Commutative law (logic)} \\
 (x \notin A) \wedge (x \in U) \Leftrightarrow & \text{Moving negation inward} \\
 x \in \bar{A} & \text{Definition of set absolute complement}
 \end{array}$$

We showed that $\forall x, x \in A \oplus U \Leftrightarrow x \in \bar{A}$. Thus, $A \oplus U = \bar{A}$. \square

- Suppose that $A \cup B = \emptyset$, what can you conclude?

Answer: we conclude that $(A = \emptyset) \wedge (B = \emptyset)$.

We formally prove that:

$$A \cup B = \emptyset \Leftrightarrow (A = \emptyset) \wedge (B = \emptyset)$$

First we consider prove the following statement:

$$A \cup B = \emptyset \rightarrow (A = \emptyset) \wedge (B = \emptyset)$$

The proof is by contradiction. We assume the antecedent and negate the conclusion.

- (1) $A \cup B = \emptyset$ given
- (2) $\neg((A = \emptyset) \wedge (B = \emptyset))$ negating the conclusion
- (3) $(A \neq \emptyset) \vee (B \neq \emptyset)$ moving negation inward
- (4) $(A \neq \emptyset) \vee (B \neq \emptyset)$ moving negation inward

We continue the proof using a proof by cases. Expression (4) states that, at least one of the following cases must hold and both can also hold:

- (a) There is at least an element in A , assume we have $x \in A$
- (b) There is at least an element in B , assume we have $y \in B$

Case (a) above: $x \in A \Rightarrow x \in \{A \cup B\}$ by definition of set union $A \cup B \neq \emptyset$, which contradicts the premise (1).

Case (b) can be shown to yield the same contradiction (exchanging A for B in the above case).

WLOG, we can conclude that

$$A \cup B = \emptyset \Rightarrow (A = \emptyset) \wedge (B = \emptyset)$$

Proving the implication in the opposite direction is straightforward by definition of set union:

$$(A = \emptyset) \wedge (B = \emptyset) \Rightarrow A \cup B = \emptyset$$

□

- Now let's look at functions, say we have the following function: $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = \lfloor \frac{x}{2} \rfloor$
 - First what does the graph of this function look like?
 - is f one-to-one (i.e., injective)? No, for example both 1 and 1.1 are assigned 0.
 - Is f onto \mathbb{R} (i.e., surjective)? No, the floor function only maps to integers, so only integers would be mapped to.
- Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $C = \{2, 7, 10\}$

Consider the following two functions: $g : A \rightarrow B$ and $f : B \rightarrow C$ where $g : \{(1, b), (2, a), (3, a), (4, b)\}$ and $f : \{(a, 10), (b, 7), (c, 2)\}$

 - Find $f \circ g$. Answer: $\{(1, 7), (2, 10), (3, 10), (4, 7)\}$
 - Find f^{-1} . Answer: $\{(10, a), (7, b), (2, c)\}$
 - Is g^{-1} a function? Answer: No, because a has two pre-images but in a function, each element of the domain must be mapped to *exactly one* element in the co-domain.

- Find $f \circ f^{-1}$. Answer: $\{(10, 10), (7, 7), (2, 2)\}$
- Prove or disprove: $\forall x, y \in \mathbb{R}, \lfloor x \times y \rfloor \leq \lfloor x \rfloor \times \lfloor y \rfloor$
 - let $x = 3.5$, and $y = 1.5$. $\lfloor 3.5 \times 1.5 \rfloor = \lfloor 5.25 \rfloor = 5$, but $\lfloor 3.5 \rfloor \times \lfloor 1.5 \rfloor = 4$
 - $5 \neq 4$, therefore the statement does not hold. This is a proof with a counterexample.
- Prove or disprove for all $x, y \in \mathbb{R}, \lceil x \times y \rceil \leq \lceil x \rceil \times \lceil y \rceil$
 - Here the same example works $\lceil 3.5 \times 1.5 \rceil = \lceil 5.25 \rceil = 6$, but $\lceil 3.5 \rceil \times \lceil 1.5 \rceil = 4 \times 2 = 8$
- Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ where $f(x) = |x|$ is not invertible, but if the domain is restricted to the set of nonnegative real numbers (i.e., $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$), the resulting function is invertible.

For a function to be invertible, it must be bijective (i.e., a one-to-one correspondence).

Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

1. Injective: No, $f(x_1) \neq f(x_2) \Rightarrow |x_1| \neq |x_2| \Rightarrow \pm x_1 \neq \pm x_2$
 Now, if the domain is restricted to the set of nonnegative real numbers. Is $f(x)$ injective?
 $f(x_1) = f(x_2) \Rightarrow |x_1| = |x_2| \Rightarrow x_1 = x_2$. Therefore, on the restricted domain $f(x)$ is injective.
 2. Surjective: Every element in codomain (\mathbb{R}^+) is a positive number, then for $\forall b \in \text{codomain}(f) \exists a \in \mathbb{R} b = |a|$. Thus, b has necessarily a preimage. Thus, the range and the codomain are equal, we can conclude that f is surjective.
 3. Bijective: No, because it is not injective.
 However, on the restricted domain, it is bijective because it is both injective and surjective.
 4. Invertible: Again, only on the restricted domain.
- Now a quick review of membership, determine whether these statements are true or false:
 1. $\{a, b\} \subseteq \{\{a, b\}\}$
 False, because neither a nor b is an element in $\{\{a, b\}\}$.
 2. $\{a, b\} \in \{\{a, b\}\}$
 True, because there the element $\{a, b\}$ is in $\{\{a, b\}\}$
 3. $\{a, b, c\} \subset \{a, b, c\}$
 False, because the sets are equal, and the statement is wondering if it is a strict subset.

4. $\{a, b, c\} \subseteq \{a, b, c\}$

True, because the sets are equal.

5. $\{\} \subseteq \{a, b, c\}$

True, because the empty set $\emptyset = \{\}$ is a subset of all sets.

6. $\emptyset \in \{a, b, c\}$

False, because the element \emptyset is not in the set $\{a, b, c\}$.

7. $\{a\} \subset \{a, a\}$

Trick question: watch out!

False, the set $\{a, a\}$ is really $\{a\}$ because, in a set, elements are *not* repeated. Therefore, $\{a\} \subset \{a, a\}$ is *false* because $\{a\} \not\subset \{a\}$ (Note that $\{a\} \subseteq \{a\}$ though).