Recitation 11: Asymptotics and Summations

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- Problem 3.2:25: Give a good big-O estimate for the following functions:
  - $(n^2+8)(n+1) = n^3+n^2+8n+1$. We can find n_0, c to prove that it is $O(n^3)$
  - $(n \log n + n^2)(n^3+1) = n^5 + n^4 \log n + n^2 + n \log n$, again easily this is $O(n^5)$, perhaps using the limit method.
  - $(n! + 2^n)(n^3 + \log(n^2+1)) = n! \cdot n^3 + n! \cdot \log(n^2+1) + 2^n \cdot n^3 + 2^n \cdot \log(n^2+1)$. Well, this one is a little bit trickier. It is actually $O(n! \cdot n^3)$, why is it not $O(n!)$ or 2^n?
- Problem 3.2:31: Show that

$$f(x) \in \Theta(g(x)) \Leftrightarrow f(x) \in O(g(x)) \wedge g(x) \in O(f(x))$$

  1. $\Rightarrow$ First we begin with a definition: $f(x) \in \Theta(g(x))$ if $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$
     - So we can see that $\exists c_1, c_2 \in (R)^+$ such that $f(x) \leq c_1 \cdot g(x)$. We also know that $f(x) \geq c_2 \cdot g(x)$.
     - Reversing the second inequality we get $g(x) \leq c \cdot f(x)$ where $c = \frac{1}{c_2}$.
     - So then we have $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$
  2. $\Leftarrow$
     - Suppose $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$.
     - Then we know that $\exists c_1, c_2$ such that $f(x) \leq c_1 \cdot g(x)$ and $g(x) \leq c_2 \cdot f(x)$.
     - so $\frac{1}{c_1} g(x) \leq f(x) \leq c_2 \cdot g(x)$.
     - But this is the definition of Θ, therefore $f(x) \in \Theta(g(x))$.
- What is the tightest bound we can form here:
  1. $x^2 + 3x + 5 \in \Delta(x^3)$: big-O
  2. $2^n \log(6) + n^2 \in \Delta 2^n)$: Θ
  3. $2^n \cdot n! + 2^n \log(n) \in \Delta(2^n)$: Ω

- To prove the first one of the previous questions, we use the limit method We have:

$$\lim_{x\to\infty} \frac{x^2+3x+4}{x^3} = \lim_{x\to\infty} \frac{2x+3}{3x^2} = \lim_{x\to\infty} \frac{2}{6x} = 0$$

  Therefore we can conclude that x^3 grows much faster, and so we get that $x^2+3x+4 \in O(x^3)$

- Next up Sequences: Can we name the first 4 terms of the following sequence $\{2^n + 1\}_{n=0}^{\infty}$?

  1. $a_0 = 2^0 + 1 = 2$
  2. $a_1 = 2^1 + 1 = 3$
  3. $a_2 = 2^2 + 1 = 5$
  4. $a_3 = 2^3 + 1 = 9$

- Now compute the following sum $\sum_{i=1}^{5} 6$ We have:

$$\sum_{i=1}^{5} 6 = 6 \sum_{i=1}^{5} 1 = 6 \cdot (5-1+1) = 6 \cdot 5 = 30$$

- How about the following geometric $\sum_{i=1}^{8} 3 \cdot 2^i$

  - We can recognize that this expression is the sum of a geometric progression (i.e., geometric series), which is in general given as $\sum_{i=0}^{n} ar^i$, except that here we start at 1 instead of zero, so we can simply compute it by subtracting the first term from the sum.
  - The formula for computing this is $\frac{a \cdot r^{n+1} - a}{r-1}$, here $r = 2$ and $a = 3$
  - Using the formula we can get $\frac{3 \cdot 2^9 - 3}{1}$ = 3·512 - 3 = 1533
  - However, remember we have to subtract the first term, so 1533 - $3 \cdot 2^0$ = 1530.