Modeling Sudoku as a CNF Formula

Spring 2017

CSCE 235H Introduction to Discrete Structures

URL: cse.unl.edu/~cse235h

All questions: Piazza

Sudoku

	8	9	4	1				
		6	7			1	9	3
2						7		
3	4		6				1	
			9					5
				2			5	
6	5			4			2	
7	3		1					

Rules:

- Each cell filled with a number 1…9
- Each
  - row,
  - column, and
  - 3x3 box

  contains all nine numbers (i.e., no duplicates)

Sudoku

1	7	3	2	6	9	5	8	4
5	8	9	4	1	3	6	7	2
4	2	6	7	5	8	1	9	3
2	9	1	5	8	4	7	3	6
3	4	5	6	7	2	8	1	9
8	6	7	9	3	1	2	4	5
9	1	4	8	2	6	3	5	7
6	5	8	3	4	7	9	2	1
7	3	2	1	9	5	4	6	8

Rules:

- Each cell filled with a number 1…9
- Each
  - row,
  - column, and
  - 3x3 box

contains all nine numbers (i.e., no duplicates)

Defining the Variables

- $p(i, j, n)$ asserts than the cell in row i and column j is assigned value n.

	8	9	4	1				
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3	4		6				1	
			9					5
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7	3		1					

$p(5, 8, 1)$

Assigning Numbers (1)

- One number in 1…9 in cell in row i, column j

$$p(i,j,1) \lor p(i,j,2) \lor \ldots \lor p(i,j,9)$$

$$\equiv \bigvee_{n=1}^{9} p(i,j,n)$$

- Every cell contains at least one number:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigvee_{n=1}^{9} p(i,j,n)$$

Assigning Numbers (2)

- The cell in row i column j cannot take two numbers

$$\forall x, y(x \neq y \to \neg(p(i,j,x) \wedge p(i,j,y)))$$

$$\equiv \forall x, y(x \neq y \to (\neg p(i,j,x) \vee \neg p(i,j,y)))$$

$$(x, y \in 1 \ldots 9) \wedge x \neq y \quad \equiv x, y = x+1, \ldots, 9$$

$$x = 1, y = 2, 3, 4, 5, 6, 7, 8, 9$$
$$x = 2, y = 3, 4, 5, 6, 7, 8, 9$$
$$x = 3, y = 4, 5, 6, 7, 8, 9$$
$$x = 4, y = 5, 6, 7, 8, 9$$
$$x = 5, y = 6, 7, 8, 9$$
$$x = 6, y = 7, 8, 9$$
$$x = 7, y = 8, 9$$
$$x = 8, y = 9$$

$$\bigwedge_{x=1}^{8} \bigwedge_{y=x+1}^{9} (\neg p(i,j,x) \vee \neg p(i,j,y))$$

Assigning Numbers (3)

- Each cell must contain a number:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigvee_{n=1}^{9} p(i,j,n)$$

- Every cell contains at most one number:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{x=1}^{8} \bigwedge_{y=x+1}^{9} (\neg p(i,j,x) \vee \neg p(i,j,y))$$

Restricting Rows, Columns

- Every row contains every number:

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$

- Every column contains every number:

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$

Restricting 3x3 Boxes

- Every 3x3 box contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

Redundant Clauses

- Sudoku problem can be modeled in many ways
- May involve redundant clauses that can be removed to obtain an equivalent formula
- Can be generated using inference rules on other clauses in the problem
- Redundant clauses may be useful and speed the solving process

Defining Initial Setup

- Initial setup is defined by including unary clauses containing the variables corresponding to the values in the filled cells

	8	9	4	1				
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			9					5
				2			5	
6	5			4			2	
7	3		1					

$$
\begin{array}{l}
\vdots \\
\wedge p(2,2,8) \\
\wedge p(2,3,9) \\
\wedge p(2,4,4) \\
\wedge p(2,5,1) \\
\wedge p(3,3,6) \\
\wedge \ldots
\end{array}
$$

Sudoku CNF Formula

- 729 variables
- 324 clauses with 9 literals
- 2916 clauses with 2 literals
- 3240 total clauses (+ clauses for initial setup)

Solving Sudoku

- Total number of possible assignments:

$$2^{729} = 2.824014 \times 10^{219}$$

- Testing one billion assignments a second:

$$8.94876 \times 10^{202} \text{ years}$$

- Modern SAT solvers can solve Sudoku in milliseconds by aggressively pruning the search tree