Recursion

Sections 8.1 and 8.2 of Rosen

Spring 2017

CSCE 235H Introduction to Discrete Structures (Honors)

Course web-page: cse.unl.edu/~cse235h

Questions: Piazza

Outline

- Introduction, Motivating Example
- Recurrence Relations
  - Definition, general form, initial conditions, terms
- Linear Homogeneous Recurrences
  - Form, solution, characteristic equation, characteristic polynomial, roots
  - Second order linear homogeneous recurrence
    - Double roots, solution, examples
    - Single root, example
  - General linear homogeneous recurrences: distinct roots, any multiplicity
- Linear Nonhomogenous Recurrences
- Other Methods
  - Backward substitution
  - Recurrence trees
  - Cheating with Maple

Recursive Algorithms

- A recursive algorithm is one in which objects are defined <u>in terms</u> of other objects of the same type
- Advantages:
  - Simplicity of code
  - Easy to understand
- Disadvantages
  - Memory
  - Speed
  - Possibly redundant work
- Tail recursion offers a solution to the memory problem, but really, do we <u>need</u> recursion?

Recursive Algorithms: Analysis

- We have already discussed how to analyze the running time of (iterative) algorithms
- To analyze recursive algorithms, we require more sophisticated techniques
- Specifically, we study how to defined & solve <u>recurrence relations</u>

Motivating Examples: Factorial

- Recall the factorial function:

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ n.(n-1) & \text{if } n > 1 \end{cases}$$

- Consider the following (recursive) algorithm for computing n!

FACTORIAL

Input: $n \in N$

Output: n!

1. **If** (n=1) or (n=0)
2. **Then Return** 1
3. **Else Return** n × FACTORIAL(n-1)
4. **Endif**
5. **End**

Factorial: Analysis

How many multiplications M(x) does factorial perform?

- When n=1 we don't perform any
- Otherwise, we perform one…
- … <u>plus</u> how ever many multiplications we perform in the recursive call FACTORIAL(n-1)
- The number of multiplications can be expressed as a formula (similar to the definition of n!

$$M(0) = 0$$

$$M(n) = 1 + M(n-1)$$

- This relation is known as a <u>recurrence relation</u>

Recurrence Relations

- **Definition**: A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms in the sequence:

$$a_0, a_1, a_2, \ldots, a_{n-1}$$

  for all integers $n \geq n_0$ where n_0 is a nonnegative integer.

- A sequence is called a solution of a recurrence if its terms satisfy the recurrence relation

Recurrence Relations: Solutions

- Consider the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$
- It has the following sequences a_n as solutions
  - $a_n = 3n$
  - $a_n = n+1$
  - $a_n = 5$
- The initial conditions + recurrence relation <u>uniquely</u> determine the sequence

Recurrence Relations: Example

- The Fibonacci numbers are defined by the recurrence

  F(n) = F(n-1) +F(n-2)

  F(1) = 1

  F(0) = 1

- The solution to the Fibonacci recurrence is

$$f_n = \frac{1}{\sqrt(5)}(\frac{(1+\sqrt(5)}{2})^n - \frac{1}{\sqrt(5)}(\frac{(1-\sqrt(5)}{2})^n$$

(The solution is derived in your textbook.)

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Recurrence Relations: General Form

- More generally, recurrences can have the form

$$T(n) = \alpha T(n-\beta) + f(n),\ T(\delta) = c$$

or

$$T(n) = \alpha T(n/\beta) + f(n),\ T(\delta) = c$$

- Note that it may be necessary to define several $T(\delta)$, which are the <u>initial conditions</u>

Recurrence Relations: Initial Conditions

- The initial conditions specify the value of the first few necessary terms in the sequence
- In the Fibonacci numbers, we needed two initial conditions:

  $$F(0)=F(1)=1$$

  because F(n) is defined by the two previous terms in the sequence
- Initial conditions are also known as <u>boundary conditions</u> (as opposed to <u>general conditions</u>)
- From now on, we will use the subscript notation, so the Fibonacci numbers are:

$$f_n = f_{n-1} + f_{n-2}$$
$$f_1 = 1$$
$$f_0 = 1$$

Recurrence Relations: Terms

- Recurrence relations have two parts:
  - recursive terms and
  - non-recursive terms

$$T(n) = 2T(n-2) + n^2 - 10$$

- Recursive terms come from when an algorithms calls itself
- Non-recursive terms correspond to the non-recursive cost of the algorithm: work the algorithm performs within a function
- We will see examples later. First, we need to know how to <u>solve</u> recurrences.

Solving Recurrences

- There are several methods for solving recurrences
  - Characteristic Equations
  - Forward Substitution
  - Backward Substitution
  - Recurrence Trees
  - … Maple!

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Linear Homogeneous Recurrences

- **Definition**: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

  with $c_1, c_2, \dots, c_k \in R$, $c_k \neq 0$.
- Linear: RHS is a sum of multiples of previous terms of the sequence (linear combination of previous terms). The coefficients are all constants (not functions depending on n)
- Homogeneous: no terms occur that are not multiples of a_j's
- Degree k: a_n is expressed in terms of $(n-k)^{th}$ term of the sequence

Linear Homogeneous Recurrences: Examples

- The Fibonacci sequence is a linear homogeneous recurrence relation
- So are the following relations:

$$a_n = 4a_{n-1} + 5a_{n-2} + 7a_{n-3}$$

$$a_n = 2a_{n-2} + 4a_{n-4} + 8a_{n-8}$$

  How many initial conditions do we need to specify for these relations?

  As many as the degree k: k = 3, 8 respectively

- So, how do <u>solve</u> linear homogeneous recurrences?

Solving Linear Homogeneous Recurrences

- We want a solution of the form $a_n = r^n$ where r is some real constant
- We observe that $a_n = r^n$ is a solution to a linear homogeneous recurrence *if and only if*

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

- We can now divide both sides by r^{n-k}, collect terms and we get a k-degree polynomial

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

- This equation is called the characteristic equation of the recurrence relation
- The roots of this polynomial are called the characteristics roots of the recurrence relation. They can be used to find the solutions (if they exist) to the recurrence relation. We will consider several cases.

Second Order Linear Homogeneous Recurrences

- A <u>second order</u> (k=2) linear homogeneous recurrence is a recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

- **Theorem** (Theorem 1, page 462): Let $c_1, c_2 \in R$ and suppose that $r^2 - c_1 r - c_2 = 0$ is the characteristic polynomial of a 2^{nd} order linear homogeneous recurrence that has two distinct* roots r_1, r_2, then $\{a_n\}$ is a solution if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

  for n=0,1,2,… where α_1, α_2 are constants dependent upon the initial conditions

* We discuss single root later

Second Order Linear Homogeneous Recurrences: Example A (1)

- Find a solution to

$$a_n = 5a_{n-1} - 6a_{n-2}$$

  with initial conditions $a_0=1$, $a_1=4$
- The characteristic equation is

$$r^2 - 5r + 6 = 0$$

- The roots are $r_1=2$, $r_2=3$

$$r^2 - 5r + 6 = (r-2)(r-3)$$

- Using the 2nd order theorem we have a solution

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

Second Order Linear Homogeneous Recurrences: Example A (2)

- Given the solution

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$

- We plug in the two initial conditions to get a system of linear equations

$$a_0 = \alpha_1 2^0 + \alpha_2 3^0$$
$$a_1 = \alpha_1 2^1 + \alpha_2 3^1$$

- Thus:

$$1 = \alpha_1 + \alpha_2$$
$$4 = 2\alpha_1 + 3\alpha_2$$

Second Order Linear Homogeneous Recurrences: Example A (3)

$$1 = \alpha_1 + \alpha_2$$
$$4 = 2\alpha_1 + 3\alpha_2$$

- Solving for $\alpha_1 = (1 - \alpha_2)$, we get

$$4 = 2\alpha_1 + 3\alpha_2$$
$$4 = 2(1-\alpha_2) + 3\alpha_2$$
$$4 = 2 - 2\alpha_2 + 3\alpha_2$$
$$2 = \alpha_2$$

- Substituting for α_1: $\alpha_1 = -1$
- Putting it back together, we have

$$a_n = \alpha_1 2^n + \alpha_2 3^n$$
$$a_n = -1\cdot 2^n + 2\cdot 3^n$$

Second Order Linear Homogeneous Recurrences: Example B (1)

- Solve the recurrence

$$a_n = -2a_{n-1} + 15a_{n-2}$$

  with initial conditions $a_0 = 0$, $a_1 = 1$

- If we did it right, we have

$$a_n = 1/8\,(3)^n - 1/8\,(-5)^n$$

- To check ourselves, we verify a_0, a_1, we compute a_3 with both equations, then maybe a_4, etc.

Single Root Case

- We can apply the theorem if the roots are distincts, i.e. $r_1 \neq r_2$
- If the roots are not distinct ($r_1 = r_2$), we say that one characteristic root has multiplicity two. In this case, we apply a different theorem
- **Theorem** (Theorem2, page 464)

  Let $c_1, c_2 \in R$ and suppose that $r^2 - c_1 r - c_2 = 0$ has only one distinct root, r_0, then $\{a_n\}$ is a solution to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 \mathbf{n} r_0^n$$

  for n=0,1,2,... where α_1, α_2 are constants depending upon the initial conditions

Single Root Case: Example (1)

- What is the solution to the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 1$, $a_1 = 7$?

- The characteristic equation is:

$$r^2 - 8r + 16 = 0$$

- Factoring gives us:

$$r^2 - 8r + 16 = (r-4)(r-4), \text{ so } r_0 = 4$$

- Applying the theorem we have the solution:

$$a_n = \alpha_1(4)^n + \alpha_2 n(4)^n$$

Single Root Case: Example (2)

- Given: $a_n = \alpha_1(4)^n + \alpha_2 n(4)^n$
- Using the initial conditions, we get:

$$a_0 = 1 = \alpha_1(4)^0 + \alpha_2 0(4)^0 = \alpha_1$$

$$a_1 = 7 = \alpha_1(4) + \alpha_2 1(4)^1 = 4\alpha_1 + 4\alpha_2$$

- Thus: $= \alpha_1 = 1$, $\alpha_2 = 3/4$
- The solution is

$$a_n = (4)^n + ¾\ n\ (4)^n$$

- Always check yourself…

General Linear Homogeneous Recurrences

- There is a straightforward generalization of these cases to higher-order linear homogeneous recurrences
- Essentially, we simply define higher degree polynomials
- The roots of these polynomials lead to a general solution
- The general solution contains coefficients that depend only on the initial conditions
- In the general case, the coefficients form a system of linear equalities

General Linear Homogeneous Recurrences: Distinct Roots

- **Theorem** (Theorem 3, page 465)

  Let $c_1, c_2, .., c_k \in R$ and suppose that the characteristic equation

  $$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

  has <u>k distinct roots</u> $r_1, r_2, \dots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

  if and only if

  $$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

  for $n=0,1,2,\dots$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants depending upon the initial conditions

General Linear Homogeneous Recurrences: Any Multiplicity

- **Theorem** (Theorem 3, page 465)

  Let $c_1, c_2, .., c_k \in R$ and suppose that the characteristic equation

  $$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

  has t roots with multiplicities $m_1, m_2, \dots, m_t$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

  if and only if $a_n = (\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m1-1} n^{m1-1}) r_1^n +$

  $$(\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m2-1} n^{m2-1}) r_2^n + \dots$$

  $$(\alpha_{t,0} + \alpha_{t,1} n + \dots + \alpha_{t,mt-1} n^{mt-1}) r_t^n$$

  for n=0,1,2,… where $\alpha_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$ depending upon the initial conditions

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Linear NonHomogeneous Recurrences

- For recursive algorithms, cost function are often not homogeneous because there is usually a non-recursive cost depending on the input size
- Such a recurrence relation is called a linear <u>non</u>homogeneous recurrence relation
- Such functions are of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \mathbf{+ f(n)}$$

- f(n) represents a non-recursive cost. If we chop it off, we are left with

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  which is the <u>associated homogeneous recurrence relation</u>
- Every solution of a linear nonhomogeneous recurrence relation is the sum of
  - a particular solution and
  - a solution to the associated linear homogeneous recurrence relation

Solving Linear NonHomogeneous Recurrences (1)

- **Theorem** (Theorem 5, p468)

  If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

  $$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} \mathbf{+ f(n)}$$

  then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

  $$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

Solving Linear NonHomogeneous Recurrences (2)

- There is no general method for solving such relations.
- However, we can solve them for special cases
- In particular, if f(n) is
  - a polynomial function
  - exponential function, or
  - the product of a polynomial and exponential functions,

then there is a general solution

Solving Linear NonHomogeneous Recurrences (3)

- **Theorem** (Theorem 6, p469)

  Suppose $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \mathbf{+ f(n)}$$

  where $c_1, c_2, .., c_k \in R$ and

  $$f(n) = (b_t n^t + b_{t-1} n^{t-1} + .. + b_1 n + b_0)\, s^n$$

  where $b_0, b_1, .., b_n, s \in R$

… continues

Solving Linear NonHomogeneous Recurrences (4)

- **Theorem** (Theorem 6, p469)… continued

  When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

  $$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

  When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

  $$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

Linear NonHomogeneous Recurrences: Examples

- The examples in the textbook are quite good (see pp467—470) and illustrate how to solve simple nonhomogeneous relations
- We may go over more examples if time allows
- Also read up on generating functions in Section 7.4 (though we may return to this subject)
- However, there are alternate, more intuitive methods

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Other Methods

- When analyzing algorithms, linear homogeneous recurrences of order greater than 2 hardly ever arise in practice
- We briefly describe two unfolding methods that work for a lot of cases
  - <u>Backward substitution</u>: this works exactly as its name suggests. Starting from the equation itself, work backwards, substituting values of the function for previous ones
  - <u>Recurrence trees</u>: just as powerful, but perhaps more intuitive, this method involves mapping out the recurrence tree for an equation. Starting from the equation, you unfold each recursive call to the function and calculate the non-recursive cost at each level of the tree. Then, you find a general formula for each level and take a summation over all such levels

Backward Substitution: Example (1)

- Give a solution to

$$T(n) = T(n-1) + 2n$$

  where $T(1)=5$

- We begin by unfolding the <u>recursion</u> by a simple substitution of the function values

- We observe that

$$T(n-1) = T((n-1) - 1) + 2(n-1) = T(n-2) + 2(n-1)$$

- Substituting into the original equation

$$T(n)=T(n-2)+2(n-1)+2n$$

Backward Substitution: Example (2)

- If we continue to do that we get

  $T(n) = T(n\text{-}2) + 2(n\text{-}1) + 2n$

  $T(n) = T(n\text{-}3) + 2(n\text{-}2) + 2(n\text{-}1) + 2n$

  $T(n) = T(n\text{-}4) + 2(n\text{-}3) + 2(n\text{-}2) + 2(n\text{-}1) + 2n$

  .....

  $T(n) = T(n\text{-}i) + \sum_{j=0}^{i-1} 2(n - j)$ *function's value at the i^{th} iteration*

- Solving the sum we get

  $T(n) = T(n\text{-}i) + 2n(i\text{-}1) – 2(i\text{-}1)(i\text{-}1\text{+}1)/2 + 2n$

  $T(n) = T(n\text{-}i) + 2n(i\text{-}1) – i^2 + i + 2n$

Backward Substitution: Example (3)

- We want to get rid of the recursive term

  $T(n) = T(n-i) + 2n(i-1) – i^2 + i + 2n$
- To do that, we need to know at what iteration we reach our based case, i.e. for what value of i can we use the initial condition $T(1)=5$?
- We get the base case when $n-i=1$ or $i=n-1$
- Substituting in the equation above we get

  $T(n) = 5 + 2n(n-1-1) – (n-1)^2 + (n-1) + 2n$

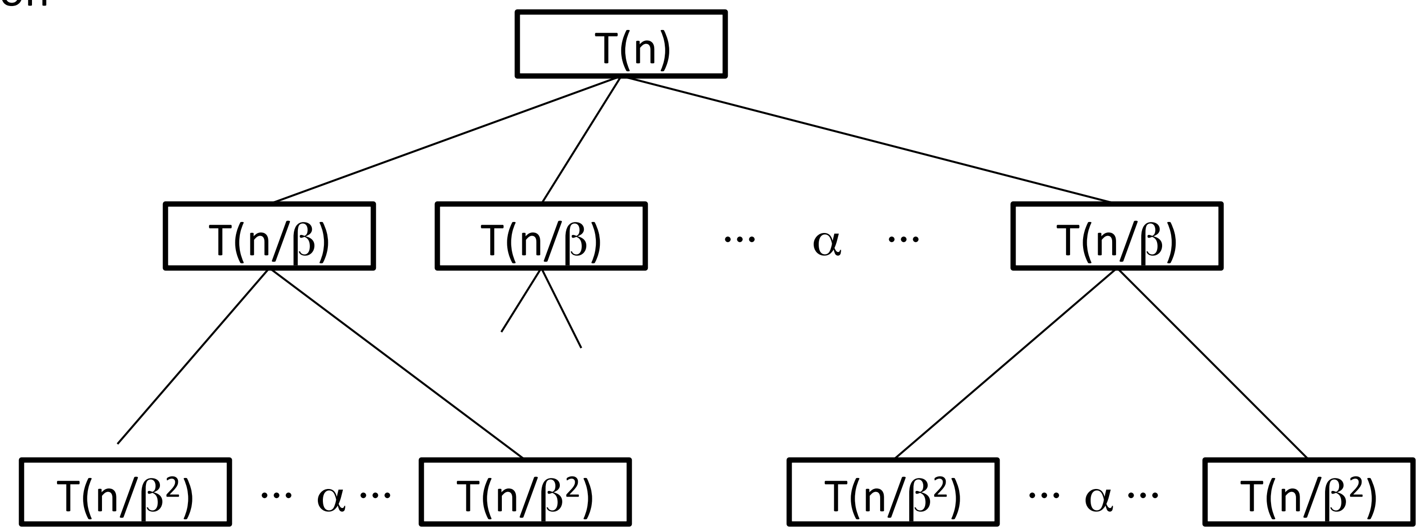
  $T(n) = 5 + 2n(n-2) – (n^2-2n+1) + (n-1) + 2n = n^2 + n + 3$

Recurrence Trees (1)

- When using recurrence trees, we graphically represent the recursion
- Each node in the tree is an instance of the function. As we progress downward, the size of the input decreases
- The contribution of each level to the function is equivalent to the number of nodes at that level times the non-recursive cost on the size of the input at that level
- The tree ends at the depth at which we reach the base case
- As an example, we consider a recursive function of the form

$$T(n) = \alpha T(n/\beta) + f(n), \quad T(\delta) = c$$

Recurrence Trees (2)



Iteration	Cost
0	$f(n)$
1	$\alpha f(n/\beta)$
2	$\alpha^2 f(n/\beta^2)$
.	.
.	.
i	$\alpha^i f(n/\beta^i)$
.	.
.	.
$\log_\beta n$	$\alpha^{\log_\beta n} \cdot T(\delta)$

Recurrence Trees (3)

- The total value of the function is the summation over all levels of the tree

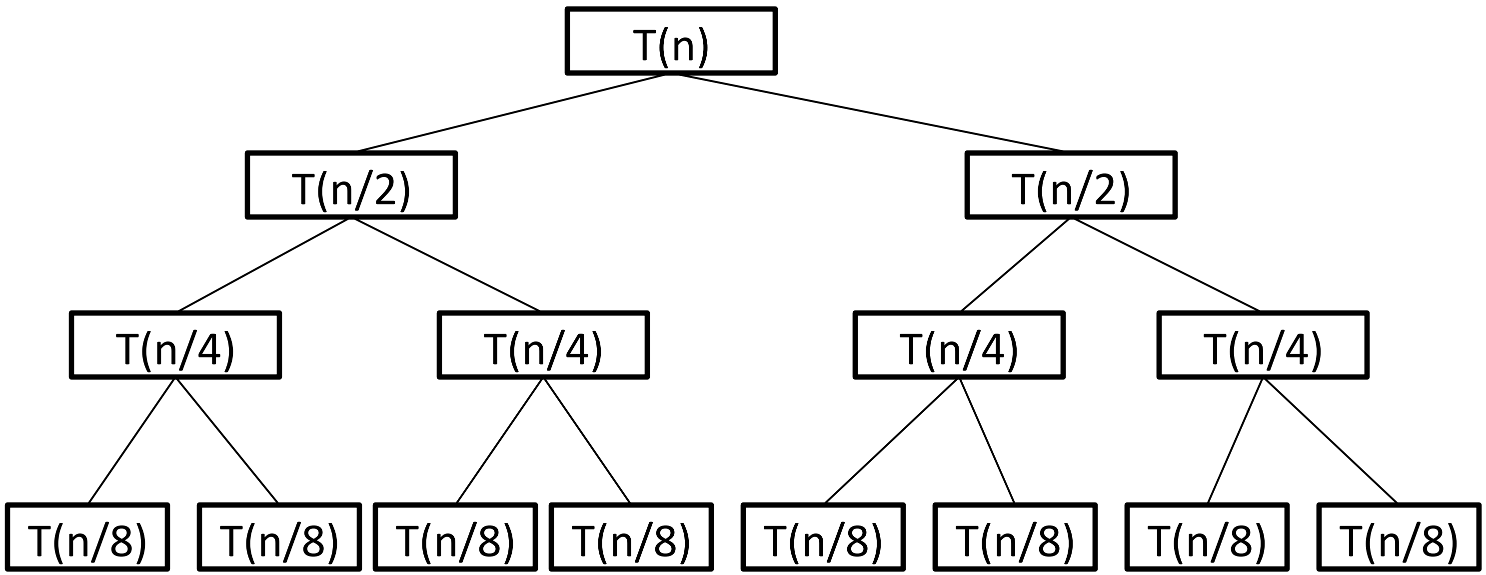
$$T(n) = \Sigma_{i=0}^{\log_\beta n} \alpha^i f(n/\beta^i)$$

$$T(n) = \alpha^{\log_\beta n} T(\delta) + \sum_{i=0}^{\log_\beta n - 1} \alpha^i f(\frac{n}{\beta^i})$$

- Consider the following concrete example

T(n) = 2T(n/2) + n,   T(1)= 4

Recurrence Tree: Example (2)



Iteration	Cost
0	n
1	n/2 +n/2
	4. n/4
2	8.n/8
.	.
.	.
i	$2^i(n/2^i)$
.	.
.	.
$\log_2 n$	$2^{\log_2 n} \cdot T(1)$

Recurrence Trees: Example (3)

- The value of the function is the summation of the value of all levels.
- We treat the last level as a special case since its non-recursive cost is different

$$T(n) = 4n + \Sigma_{i=0}^{(\log_2 n - 1)} 2^i \frac{n}{2^i} = n(\log n) + 4n$$

Smoothness Rule

- In the previous example, we make the following assumption

  n has a power of two ($n=2^k$)

  This assumption is necessary to get a nice depth of log(n) and a full tree
- We can restrict consideration to certain powers because of the smoothness rule, which is <u>not</u> studied in this course.
- For more information about that rule, consult pages 481—483 of the textbook "The Design & Analysis of Algorithms" by Anany Levitin

How to Cheat with Maple (1)

- Maple and other math tools are great resources. However, they are no substitutes for knowing how to solve recurrences yourself
- As such, you should only use Maple to check you answers
- Recurrence relations can be solved using the `rsolve` command and giving Maple the proper parameters
- The arguments are essentially a comma-delimited list of equations
  - General and boundary conditions
  - Followed by the 'name' and variables of the function

How to Cheat with Maple (2)

```
> rsolve({T(n)= T(n-1)+2*n,T(1)=5},T(n));
```

1+2(n+1)(1/2n+1)-2n

- You can clean up Maple's answer a bit by encapsulating it in the `simplify` command

```
> simplify(rsolve({T(n)= T(n-1) + 2*n, T(1) = 5},
  T(n)));
```

$3 + n^2 + n$

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