Recitation 3

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- Questions about Piazza, LaTeX or lecture?
- Questions on the homework?
- Go over quiz from last week (max of 3 mins).
- Go over last week's homework (max of 3 mins).
- Let's start by looking at section 1.3, problem 63 on page 36 of your Rosen textbook: "Show how the solution of a given 4X4 Sudoku puzzle can be solved by solving a satisfiability problem (SAT).

  1. First, let's construct the predicate $P(x, y, n)$, which means row x, column y contains integer n.
  2. Since we are talking about 4x4 Sudoku puzzles, we have 4x4x4 = 64 total propositions.
  3. Our goal is to find a truth assignment to each of these 64 propositions that solves the Sudoku. To do this, we need to make a few assertions:
     - For each cell with a given value, we assert $P(x, y, n)$ when the cell in row x and column y has value n.
     - We assert that every row contains every number: $\bigwedge_{x=1}^{4} \bigwedge_{n=1}^{4} \bigvee_{y=1}^{4} P(x, y, n)$.
     - We assert that every row contains every number: $\bigwedge_{y=1}^{4} \bigwedge_{n=1}^{4} \bigvee_{x=1}^{4} P(x, y, n)$.
     - We assert that each of the 2x2 blocks contains every number: $\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{4} \bigvee_{x=1}^{2} \bigvee_{y=1}^{2} P(2r + x, 2s + y, n)$
     - We assert that no cell contains more than one number. To do this, we take the conjunction of all values of x,y,n, and $n\prime$ for 1 to 4, where $n \neq n\prime$, of $P(x, y, n) \to \neg P(x, y, n\prime)$.
  4. Now to construct the assertion that row x contains every number n: $\bigvee_{y=1}^{4} P(x, y, n)$.
  5. To construct the assertion that all rows contain every number n: $\bigwedge_{n=1}^{4} \bigvee_{y=1}^{4} P(x, y, n)$.
  6. Building on that to include every column: $\bigwedge_{x=1}^{4} \bigwedge_{n=1}^{4} \bigvee_{y=1}^{4} P(x, y, n)$

  - What is an example of a term from this SAT instance? $P(1, 3, 2)$
  - What is an example of a literal from this SAT instance? $P(1, 3, 2) or \neg P(1, 3, 2)$
  - What is an example of a clause? $(P(1, 1, 1) \vee P(1, 2, 1) \vee (P(1, 3, 1) \vee P(1, 4, 1))$

* Notice that this clause says that at least one column contains the number 1.

- politicians can fool some of the people all of the time, all of the people some of the time, but they cannot fool all of the people all of the time.

  1. Predicates and their meaning:
     - $Fools(x, y, t)$: x fool y at time t.
     - $P(x)$: x is a politician.
  2. Universe of discourse: x, y all human beings, t all time instants.
  3. $\forall x[P(x) \rightarrow [(\exists y \forall t Fools(x, y, t)) \wedge (\exists t \forall y Fools(x, y, t)) \wedge \neg(\forall y \forall t Fools(x, y, t))]]$

  The expression $\neg(\forall y \forall t Fools(x, y, t))$ can be difficult to spell out in English. We recommend that you use the expression "*it is not the case that*" whenever an expression starts with a negation. Thus, the above reads: "It is not the case that every x fools every y at every time t."

- Beware of errors:

  - Let's compare the meaning of the two expressions:
    * $\exists t \forall y Fools(x, y, t)$: There is at least one time (or one incident) where everyone was fooled by x.
    * $\forall y \exists t Fools(x, y, t)$: Everyone is fooled at some point in time by x, but the time is not necessarily the same for all human beings.

    Naturally, we intend the first meaning.
  - Also, compare the meaning of the following two expressions:
    * $\forall x[P(x) \rightarrow \neg(\forall y \forall t Fools(x, y, t))] \equiv \forall x[P(x) \rightarrow (\exists y \exists t \neg Fools(x, y, t))]$: It is not the case that every x fools every y at every time t.
    * $\forall x[P(x) \rightarrow (\forall y \forall t \neg Fools(x, y, t))]$: Politicians do not ever fool anyone.

- Now let's look at Rosen 1.4:43 Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

  1. First, do we think they are equivalent? **No**.
  2. Let $P(x)$ be a proposition that is sometimes true and sometimes false. Then let $Q(x)$ be some proposition that is always false. For example let $P(x)$ be x is greater than 5,000, and $Q(x)$ be x is less than zero. And let our universe of discourse be all integers greater than or equal to zero.
  3. $\forall x(P(x) \rightarrow Q(x))$ is false and $\forall x P(x) \rightarrow \forall x Q(x)$ is false.
     - Notice this is because $\forall x(P(x)$ can be false, and $\forall Q(x))$ is always false. So $\forall x(P(x) \rightarrow Q(x))$ is not going to be true, because for some x, $P(x)$ is true, so $P(x) \rightarrow Q(x)$ is sometimes false. Therefore the entire statement is false.

  - Whereas $\forall x P(x) \rightarrow \forall x Q(x)$ is true because both $\forall x P(x)$ and $\forall x Q(x)$ are false.

- Now let's look at Rosen 1.4:41 part a. Express the following using predicates, quantifiers and logical connectives:

  - *At least one mail message, among a nonempty set of messages, can be saved if there is a disk with more than ten kilobits of free space.*

  1. First let $D(x, y)$ be the statement *Disk x has more than y kilobits free.* Let $S(x)$ be the statement *The message x can be saved.*
  2. We can express the specification in the following way: $\exists x D(x, 10) \rightarrow \exists z S(z)$.

- Next, Rosen 1.5:31 part d. Express the negation of the following statement so that negation immediately precedes a predicate.

  - $\forall x \exists y (P(x, y) \rightarrow Q(x, y))$
    1. First negate: $\neg \forall x \exists y (P(x, y) \rightarrow Q(x, y))$
    2. Move the negation inside quantifiers: $\exists x \forall y \neg (P(x, y) \rightarrow Q(x, y))$
    3. Next get rid of the implication: $\exists x \forall y \neg (\neg P(x, y) \vee Q(x, y))$
    4. Now distribute the negation through the parentheses:

$$\exists x \forall y (P(x, y) \wedge \neg Q(x, y))$$

  - Done!

- Rosen 1.3:15
  Show that $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology

Step	**Sentence**	**Equivalence law**
0.	$\neg(\neg q \wedge (p \rightarrow q)) \vee \neg p$	
1.	$\equiv \neg(\neg q \wedge (\neg p \vee q)) \vee \neg p$	Implication law
2.	$\equiv (q \vee (p \wedge \neg q)) \vee \neg p$	DeMorgan's law
3.	$\equiv ((q \vee p) \wedge (q \vee \neg q)) \vee \neg p$	Distributive law
4.	$\equiv ((q \vee p)) \vee \neg p$	Identity law
5.	$\equiv (q \vee (p \vee \neg p))$	Associative law
6.	$\equiv q \vee T$	Identity law
7.	$\equiv T$	Identity law

- Rosen 1.3:23
  Show that $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

Step	**Sentence**	**Equivalence law**
0.	$(p \rightarrow r) \wedge (q \rightarrow r)$	
1.	$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$	Implication law
2.	$\equiv (\neg p \wedge \neg q) \vee r$	Distributive law
3.	$\equiv \neg(\neg p \wedge \neg q) \rightarrow r$	Implication law
4.	$\equiv (p \vee q) \rightarrow r$	DeMorgan's law and double negation law

- (Last 10 minutes) Give quiz