Recitation 10

Created by Taylor Spangler, Adapted by Beau Christ

• (3 min max) Questions about last week’s quiz?

• Questions about lecture / homework so far?

• Now, problem 5.1.5, using induction prove: $1^2 + 3^2 + \ldots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for nonnegative $n$.

1. Here we can see the base case is 0 (we want $n$ to be nonnegative and an integer, note not the same as positive), what is $P(0)$? $P(0) = 1 = \frac{(1)(1)(3)}{3}$
2. Show that $P(0)$ is true: $1 = 1$. Therefore $P(0)$ is true.
3. What is the inductive hypothesis? $P(k)$ is true, that is $1^2 + 3^2 + \ldots + (2k + 1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.
4. We want to prove: $P(k+1)$ that is $1^2 + 3^2 + \ldots + (2k + 3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$
5. So we have that $P(k)$ is true, $P(k+1)$ is really just $1^2 + 3^2 + \ldots + (2k + 1)^2 + (2k + 3)^2$. By our inductive hypothesis, we have that this is really $\frac{(k+1)(2k+1)(2k+3)}{3} + (2k + 3)^2$. This equals $(2k + 3) * \frac{(k+1)(2k+1)+6k+9}{3} = (2k + 3) * \frac{2k^3+9k+10}{3}$...this is what we were trying to prove, therefore, by induction, $1^2 + 3^2 + \ldots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for nonnegative $n$.

• Now, prove the following: $3 \mid 2^{2n} - 1$ for $n \geq 1$.

1. Base case is $n = 1$. So $P(1) = 3 \mid 2^{2(1)} - 1$.
2. Well clearly $2^{2(1)} - 1 = 4 - 1 = 3$. And it is obvious that $3 \mid 3$, so the base case is proven.
3. What is our inductive hypothesis? $P(k)$ so $3 \mid 2^{2k} - 1$.
4. We want to prove that $3 \mid 2^{2k+2} - 1$.
5. Well, $P(k+1) = 3 \mid 2^{2k+2} - 1$. We also can see $2^{2k+2} - 1 = 4 * 2^k - 1 = 4 * (2^k - 1 + 1) - 1$.
6. But we know, by our inductive hypothesis that this equals $4 * (3t + 1) - 1 = 12t + 4 - 1 = 12t + 3 = 3(4t + 1)$.
7. Clearly this is divisible by 3, therefore $P(k+1) = 3 \mid 2^{2k+2} - 1$ is true.
• A strong induction proof.

Show that if \( n \in \mathbb{N} \) then \( 12 \mid (n^4 - n^2) \).

1. **Base Case:**
   (a) \( n = 1 : 1^4 - 1^2 = 0 = 12 * 0 \) so \( P(1) \) is true.
   (b) \( n = 2 : 2^4 - 2^2 = 16 - 4 = 12 = 12 * 1 \) so \( P(2) \) is true.
   (c) \( n = 3 : 3^4 - 3^2 = 81 - 9 = 72 = 12 * 6 \) so \( P(3) \) is true.
   (d) \( n = 4 : 4^4 - 4^2 = 256 - 16 = 240 = 12 * 20 \) so \( P(4) \) is true.
   (e) \( n = 5 : 5^4 - 5^2 = 625 - 25 = 600 = 12 * 50 \) so \( P(5) \) is true.
   (f) \( n = 6 : 6^4 - 6^2 = 1296 - 36 = 1260 = 12 * 105 \) so \( P(6) \) is true

2. **Strong Inductive Hypothesis** Let \( k \geq 6 \in \mathbb{N} \) and assume that \( 12 \mid (m^4 - m^2) \) for \( 1 \leq m \leq k \) where \( m \in \mathbb{N} \).

3. Let \( i = k - 5 \), then we can assume \( P(i) \) holds. Clearly \( i + 6 = k + 1 \).

4. \((i + 6)^4 - (i + 6)^2 = (i^4 + 24i^3 + 180i^2 + 864i + 1296) - (i^2 + 12i + 36) = (i^4 - i^2) + 24i^3 + 180i^2 + 852i + 1260\)

5. Clearly \( i^4 - i^2 = 12 * t \).

6. We can also see that the remaining portion can be rewritten as \( 12(2i^3 + 15i^2 + 71i + 105) \).

7. Then we have that \( (k + 1)^4 - (k + 1)^2 = 12t + 12(2i^3 + 15i^2 + 71i + 105) = 12 * (t + 2i^3 + 15i^2 + 71i + 105) \)

8. Then we have that \( 12 \mid (k + 1)^4 - (k + 1)^2 \).