# Recitation 10

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- (3 min max) Questions about last week's quiz?
- Questions about lecture / homework so far?
- Now, problem 5.1.5, using induction prove: $1^2 + 3^2 + ... + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for nonnegative $n$.

  1. Here we can see the base case is 0 (we want $n$ to be nonnegative and an integer, note not the same as positive), what is $P(0)$? $P(0) = 1 = \frac{(1)(1)(3)}{3}$
  2. Show that P(0) is true:$1 = 1.\frac{1*1*3}{3} = 1$. Therefore P(0) is true.
  3. What is the inductive hypothesis? $P(k)$ is true, that is $1^2 + 3^2 + ... + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$.
  4. We want to prove: $P(k+1)$ that is $1^2 + 3^2 + ... + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$
  5. So we have that $P(k)$ is true, $P(k+1)$ is really just $1^2 + 3^2 + ... + (2k+1)^2 + (2k+3)^2$. By our inductive hypothesis, we have that this is really $\frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$. This equals $(2k+3) * (\frac{(k+1)(2k+1)+6k+9}{3}) = (2k+3) * (\frac{2k^2+9k+10}{3}) = \frac{(k+2)(2k+3)(2k+5)}{3}$...this is what we were trying to prove, therefore, by induction, $1^2 + 3^2 + ... + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ for nonnegative $n$.

- Now, prove the following: $3 \mid 2^{2n} - 1$ for $n \geq 1$.

  1. Base case is $n = 1$. So $P(1) = 3 \mid 2^{2(1)} - 1$.
  2. Well clearly $2^{2(1)} - 1 = 4 - 1 = 3$. And it is obvious that $3 \mid 3$, so the base case is proven.
  3. What is our inductive hypothesis? $P(k)$ so $3 \mid 2^{2k} - 1$.
  4. We want to prove that $3 \mid 2^{2k+2} - 1$.
  5. Well, $P(k+1) = 3 \mid 2^{2k+2} - 1$. We also can see $2^{2k+2} - 1 = 4 * 2^k - 1 = 4 * (2^k - 1 + 1) - 1$.
  6. But we know, by our inductive hypothesis that this equals $4 * (3t+1) - 1 = 12t + 4 - 1 = 12t + 3 = 3(4t+1)$.
  7. Clearly this is divisible by 3, therefore $P(k+1) = 3 \mid 2^{2k+2} - 1$ is true.

- A strong induction proof.

$$\text{Show that if } n \in \mathbb{N} \text{ then } 12 \mid (n^4 - n^2).$$

1. **Base Case:**
   (a) $n = 1$: $1^4 - 1^2 = 0 = 12 * 0$ so $P(1)$ is true.
   (b) $n = 2 : 2^4 - 2^2 = 16 - 4 = 12 = 12 * 1$ so $P(2)$ is true.
   (c) $n = 3 : 3^4 - 3^2 = 81 - 9 = 72 = 12 * 6$ so $P(3)$ is true.
   (d) $n = 4 : 4^4 - 4^2 = 256 - 16 = 240 = 12 * 20$ so $P(4)$ is true.
   (e) $n = 5 : 5^4 - 5^2 = 625 - 25 = 600 = 12 * 50$ so $P(5)$ is true.
   (f) $n = 6 : 6^4 - 6^2 = 1296 - 36 = 1260 = 12 * 105$ so $P(6)$ is true
2. **Strong Inductive Hypothesis** Let $k \geq 6 \in \mathbb{N}$ and assume that $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$ where $m \in \mathbb{N}$.
3. Let $i = k - 5$, then we can assume $P(i)$ holds. Clearly $i + 6 = k + 1$.
4. $(i + 6)^4 - (i + 6)^2 = (i^4 + 24i^3 + 180i^2 + 864i + 1296) - (i^2 + 12i + 36) = (i^4 - i^2) + 24i^3 + 180i^2 + 852i + 1260$
5. Clearly $i^4 - i^2 = 12 * t$.
6. We can also see that the remaining portion can be rewritten as $12(2i^3 + 15i^2 + 71i + 105)$.
7. Then we have that $(k + 1)^4 - (k + 1)^2 = 12t + 12(2i^3 + 15i^2 + 71i + 105) = 12 * (t + 2i^3 + 15i^2 + 71i + 105)$
8. Then we have that $12 \mid (k + 1)^4 - (k + 1)^2$.