Sets

Sections 2.1 and 2.2 of Rosen

Spring 2013
CSCE 235 Introduction to Discrete Structures
Course web-page: cse.unl.edu/~cse235
Questions: Piazza
Outline

• Definitions: set, element
• Terminology and notation
  • Set equal, multi-set, bag, set builder, intension, extension, Venn Diagram (representation), empty set, singleton set, subset, proper subset, finite/infinite set, cardinality
• Proving equivalences
• Power set
• Tuples (ordered pair)
• Cartesian Product (a.k.a. Cross product), relation
• Quantifiers
• Set Operations (union, intersection, complement, difference), Disjoint sets
• Set equivalences (cheat sheet or Table 1, page 130)
  • Inclusion in both directions
  • Using membership tables
• Generalized Unions and Intersection
• Computer Representation of Sets
Introduction (1)

• We have already implicitly dealt with sets
  – Integers (\(\mathbb{Z}\)), rationals (\(\mathbb{Q}\)), naturals (\(\mathbb{N}\)), reals (\(\mathbb{R}\)), etc.

• We will develop more fully
  – The definitions of sets
  – The properties of sets
  – The operations on sets

• **Definition**: A set is an *unordered* collection of *(unique)* objects

• Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs
Introduction (2)

• **Definition:** The objects in a set are called **elements** or **members** of a set. A set is said to contain its elements.

• **Notation,** for a set A:
  - $x \in A$: $x$ is an element of $A$
  - $x \notin A$: $x$ is not an element of $A$
Terminology (1)

- **Definition:** Two sets, A and B, are equal if they contain the same elements. We write $A = B$.

- **Example:**
  - $\{2, 3, 5, 7\} = \{3, 2, 7, 5\}$, because a set is unordered.
  - Also, $\{2, 3, 5, 7\} = \{2, 2, 3, 5, 3, 7\}$ because a set contains unique elements.
  - However, $\{2, 3, 5, 7\} \neq \{2, 3\}$
Terminology (2)

• A **multi-set** is a set where you specify the number of occurrences of each element: \( \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\} \) is a set where
  – \( m_1 \) occurs \( a_1 \) times
  – \( m_2 \) occurs \( a_2 \) times
  – …
  – \( m_r \) occurs \( a_r \) times

• In Databases, we distinguish
  – A set: elements cannot be repeated
  – A **bag**: elements can be repeated
Terminology (3)

• The **set-builder** notation
  \[ O = \{ x \mid (x \in \mathbb{Z}) \land (x = 2k) \text{ for some } k \in \mathbb{Z} \} \]
  reads: \( O \) is the set that contains all \( x \) such that \( x \) is an integer and \( x \) is even

• A set is defined in **intension** when you give its set-builder notation
  \[ O = \{ x \mid (x \in \mathbb{Z}) \land (0 \leq x \leq 8) \land (x = 2k) \text{ for some } k \in \mathbb{Z} \} \]

• A set is defined in **extension** when you enumerate all the elements:
  \[ O = \{0, 2, 4, 6, 8\} \]
Venn Diagram: Example

- A set can be represented graphically using a Venn Diagram
More Terminology and Notation (1)

- A set that has no elements is called the **empty set** or **null set** and is denoted $\emptyset$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\emptyset \neq \{\emptyset\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set
More Terminology and Notation

(2)

• **Definition:** A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B.

• That is, we have the equivalence:

$$A \subseteq B \iff \forall x (x \in A \Rightarrow x \in B)$$
More Terminology and Notation (3)

- **Theorem:** For any set $S$
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$  
  
- The proof is in the book, an excellent example of a vacuous proof

*Theorem 1, page 120*
More Terminology and Notation (4)

- **Definition**: A set $A$ that is a subset of a set $B$ is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$, $A \subsetneq B$
- In LaTeX: $\subset$, $\subsetneq$
More Terminology and Notation

(5)

• Sets can be elements of other sets

• Examples
  – \( S_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, c\} \)
  – \( S_2 = \{\{1\}, \{2,4,8\}, \{3\}, \{6\}, 4, 5, 6\} \)
More Terminology and Notation (6)

• **Definition**: If there are exactly \( n \) distinct elements in a set \( S \), with \( n \) a nonnegative integer, we say that:
  
  – \( S \) is a **finite set**, and
  
  – The **cardinality** of \( S \) is \( n \). Notation: \( |S| = n \).

• **Definition**: A set that is not finite is said to be **infinite**
More Terminology and Notation (7)

• Examples
  – Let \( B = \{ x \mid (x \leq 100) \land (x \text{ is prime}) \} \), the cardinality of \( B \) is \( |B| = 25 \) because there are 25 primes less than or equal to 100.
  – The cardinality of the empty set is \( |\emptyset| = 0 \)
  – The sets \( N, Z, Q, R \) are all infinite
Proving Equivalence (1)

• You may be asked to show that a set is
  – a subset of,
  – proper subset of, or
  – equal to another set.

• To prove that $A$ is a subset of $B$, use the equivalence discussed earlier
  $A \subseteq B \iff \forall x (x \in A \implies x \in B)$
    – To prove that $A \subseteq B$ it is enough to show that for an arbitrary
      (nonspecific) element $x$, $x \in A$ implies that $x$ is also in $B$.
    – Any proof method can be used.

• To prove that $A$ is a proper subset of $B$, you must prove
  – $A$ is a subset of $B$ and
  – $\exists x \ (x \in B) \land (x \notin A)$
Proving Equivalence (2)

• Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  – $A \subseteq B$ and
  – $B \subseteq A$

• Logically speaking, you must show the following quantified statements:
  $$(\forall x (x \in A \implies x \in B)) \land (\forall x (x \in B \implies x \in A))$$
  we will see an example later.
• **Definition:** The power set of a set $S$, denoted $P(S)$, is the set of all subsets of $S$.

• **Examples**
  - Let $A=\{a,b,c\}$, $P(A)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$
  - Let $A=\{\{a,b\},c\}$, $P(A)=\{\emptyset,\{\{a,b\}\},\{c\},\{\{a,b\},c\}\}$

• **Note:** the empty set $\emptyset$ and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 120).
Power Set (2)

• The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set

• **Fact:** Let $S$ be a set such that $|S|=n$, then

$$|P(S)| = 2^n$$
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Tuples (1)

• Sometimes we need to consider ordered collections of objects

• **Definition**: The ordered n-tuple \((a_1, a_2, \ldots, a_n)\) is the ordered collection with the element \(a_i\) being the i-th element for \(i=1,2,\ldots,n\)

• Two ordered n-tuples \((a_1, a_2, \ldots, a_n)\) and \((b_1, b_2, \ldots, b_n)\) are equal iff for every \(i=1,2,\ldots,n\) we have \(a_i=b_i\)

• A 2-tuple \((n=2)\) is called an ordered pair
 Cartesian Product (1)

- **Definition:** Let $A$ and $B$ be two sets. The **Cartesian product** of $A$ and $B$, denoted $A \times B$, is the set of all ordered pairs $(a,b)$ where $a \in A$ and $b \in B$

  $$A \times B = \{(a,b) \mid (a \in A) \land (b \in B)\}$$

- The Cartesian product is also known as the **cross product**

- **Definition:** A subset of a Cartesian product, $R \subseteq A \times B$ is called a **relation**. We will talk more about relations in the next set of slides

- **Note:** $A \times B \neq B \times A$ unless $A = \emptyset$ or $B = \emptyset$ or $A = B$. Find a counter example to prove this.
Cartesian Product (2)

• Cartesian Products can be generalized for any n-tuple

• **Definition:** The Cartesian product of n sets, \( A_1, A_2, \ldots, A_n \), denoted \( A_1 \times A_2 \times \ldots \times A_n \), is
  \[ A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \} \]
Notation with Quantifiers

• Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language.

• Now we can simplify things using set notation!

• Example
  - $\forall x \in \mathbb{R} \ (x^2 \geq 0)$
  - $\exists x \in \mathbb{Z} \ (x^2 = 1)$
  - Also mixing quantifiers:
    $$\forall a, b, c \in \mathbb{R} \ \exists x \in \mathbb{C} \ (ax^2 + bx + c = 0)$$
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Set Operations

- Arithmetic operators (+, -, \times, \div) can be used on pairs of numbers to give us new numbers.
- Similarly, set operators exist and act on two sets to give us new sets:
  - Union $\cup$
  - Intersection $\cap$
  - Set difference $\setminus$
  - Set complement $\overline{S}$
  - Generalized union $\bigcup$
  - Generalized intersection $\bigcap$
Set Operators: Union

- **Definition**: The *union* of two sets $A$ and $B$ is the set that contains all elements in $A$, $B$, or both. We write:

$$A \cup B = \{ x \mid (x \in A) \lor (x \in B) \}$$
Set Operators: Intersection

- **Definition**: The intersection of two sets $A$ and $B$ is the set that contains all elements that are element of both $A$ and $B$. We write:

$$A \cap B = \{ x \mid (x \in A) \land (x \in B) \}$$
Disjoint Sets

- **Definition:** Two sets are said to be disjoint if their intersection is the empty set: \( A \cap B = \emptyset \)
Set Difference

- **Definition:** The **difference** of two sets $A$ and $B$, denoted $A \setminus B$ (\$	ext{setminus}\$) or $A - B$, is the set containing those elements that are in $A$ but not in $B$.  

![Venn Diagram](image)
Set Complement

- **Definition**: The complement of a set $A$, denoted $\overline{A}$ ($\overline{\text{bar}}$), consists of all elements **not** in $A$. That is, the difference of the universal set and $U$: $U \setminus A$

\[ \overline{A} = A^c = \{ x \mid x \notin A \} \]
Set Complement: Absolute & Relative

- Given the Universe $U$, and $A, B \subseteq U$.
- The (absolute) complement of $A$ is $A = U \setminus A$.
- The (relative) complement of $A$ in $B$ is $B \setminus A$. 

![Diagram of set complement](image-url)
Set Identities

• There are analogs of all the usual laws for set operations. Again, the Cheat Sheet is available on the course webpage: http://www.cse.unl.edu/~cse235/files/LogicalEquivalences.pdf

• Let’s take a quick look at this Cheat Sheet or at Table 1 on page 130 in your textbook
Proving Set Equivalences

• Recall that to prove such identity, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal

• The book proves several of the standard set identities

• We will give a couple of different examples here
Proving Set Equivalences: Example A (1)

• Let
  – \(A=\{x \mid x \text{ is even}\}\)
  – \(B=\{x \mid x \text{ is a multiple of 3}\}\)
  – \(C=\{x \mid x \text{ is a multiple of 6}\}\)

• Show that \(A \cap B = C\)
Proving Set Equivalences: Example A (2)

• \( A \cap B \subseteq C \): \( \forall x \in A \cap B \)
  \( \Rightarrow \) \( x \) is a multiple of 2 and \( x \) is a multiple of 3
  \( \Rightarrow \) we can write \( x = 2 \cdot 3 \cdot k \) for some integer \( k \)
  \( \Rightarrow \) \( x = 6k \) for some integer \( k \) \( \Rightarrow \) \( x \) is a multiple of 6
  \( \Rightarrow \) \( x \in C \)

• \( C \subseteq A \cap B \): \( \forall x \in C \)
  \( \Rightarrow \) \( x \) is a multiple of 6 \( \Rightarrow \) \( x = 6k \) for some integer \( k \)
  \( \Rightarrow \) \( x = 2(3k) = 3(2k) \) \( \Rightarrow \) \( x \) is a multiple of 2 and of 3
  \( \Rightarrow \) \( x \in A \cap B \)
Proving Set Equivalences: Example B (1)

• An alternative prove is to use membership tables where an entry is
  – 1 if a chosen (but fixed) element is in the set
  – 0 otherwise

• Example: Show that

\[
\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}
\]
Proving Set Equivalences: Example B (2)

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- 1 under a set indicates that “an element is in the set”
- If the columns are equivalent, we can conclude that indeed the two sets are equal
Generalizing Set Operations: Union and Intersection

• In the previous example, we showed De Morgan’s Law generalized to unions involving 3 sets
• In fact, De Morgan’s Laws hold for any finite set of sets
• Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets
Generalized Union

• **Definition:** The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

\[ \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \]

LaTeX: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$
Generalized Intersection

- **Definition:** The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection.

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \]

LaTeX: $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$
Computer Representation of Sets (1)

• There really aren’t ways to represent infinite sets by a computer since a computer has a finite amount of memory.

• If we assume that the universal set $U$ is finite, then we can easily and effectively represent sets by bit vectors.

• Specifically, we force an ordering on the objects, say:

  $U=\{a_1, a_2, \ldots, a_n\}$

• For a set $A \subseteq U$, a bit vector can be defined as, for $i=1,2,\ldots,n$
  - $b_i=0$ if $a_i \notin A$
  - $b_i=1$ if $a_i \in A$
Computer Representation of Sets (2)

• Examples
  – Let U={0,1,2,3,4,5,6,7} and A={0,1,6,7}
  – The bit vector representing A is: 1100 0011
  – How is the empty set represented?
  – How is U represented?

• Set operations become trivial when sets are represented by bit vectors
  – Union is obtained by making the bit-wise OR
  – Intersection is obtained by making the bit-wise AND
Computer Representation of Sets (3)

- Let $U=\{0,1,2,3,4,5,6,7\}$, $A=\{0,1,6,7\}$, $B=\{0,4,5\}$
- What is the bit-vector representation of $B$?
- Compute, bit-wise, the bit-vector representation of $A \cap B$
- Compute, bit-wise, the bit-vector representation of $A \cup B$
- What sets do these bit vectors represent?
Programming Question

• Using bit vector, we can represent sets of cardinality equal to the size of the vector

• What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?

• What data structure could we use?