

Partial Orders

Section 9.6 of Rosen

Spring 2013

CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Questions: Piazza

Outline

- Motivating example
- Definitions
 - Partial ordering, comparability, total ordering, well ordering
- Principle of well-ordered induction
- Lexicographic orderings
- Hasse Diagrams
- Extremal elements
- Lattices
- Topological Sorting

Motivating Example (1)

- Consider the renovation of Avery Hall. In this process several tasks were undertaken
 - Remove Asbestos
 - Replace windows
 - Paint walls
 - Refinish floors
 - Assign offices
 - Move in office furniture

Motivating Example (2)

- Clearly, some things had to be done before others could begin
 - Asbestos had to be removed before anything (except assigning offices)
 - Painting walls had to be done before refinishing floors to avoid ruining them, etc.
- On the other hand, several things could be done concurrently:
 - Painting could be done while replacing the windows
 - Assigning offices could be done at anytime before moving in office furniture
- This scenario can be nicely modeled using partial orderings

Partial Orderings: Definitions

- **Definitions:**

- A relation R on a set S is called a partial order if it is

- Reflexive
 - Antisymmetric
 - Transitive

- A set S together with a partial ordering R is called a partially ordered set (poset, for short) and is denote (S,R)

- Partial orderings are used to give an order to sets that may not have a natural one
- In our renovation example, we could define an ordering such that $(a,b) \in R$ if ‘a must be done before b can be done’

Partial Orderings: Notation

- We use the notation:
 - $a \preceq b$, when $(a,b) \in R$ $\$\\preccurlyeq\$\$
 - $a \prec b$, when $(a,b) \in R$ and $a \neq b$ $\$\\prec\$\$
- The notation \prec is not to be mistaken for “less than” (\prec versus \leq)
- The notation \prec is used to denote any partial ordering

Comparability: Definition

- **Definition:**
 - The elements a and b of a poset (S, \preceq) are called comparable if either $a \preceq b$ or $b \preceq a$.
 - When for $a, b \in S$, we have neither $a \preceq b$ nor $b \preceq a$, we say that a, b are incomparable
- Consider again our renovation example
 - Remove Asbestos $\prec a_i$ for all activities a_i except assign offices
 - Paint walls \prec Refinish floors
 - Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices

Total orders: Definition

- **Definition:**

- If (S, \preceq) is a poset and every two elements of S are comparable, S is called a totally ordered set.
- The relation \preceq is said to be a total order

- **Example**

- The relation “less than or equal to” over the set of integers (\mathbb{Z}, \leq) since for every $a, b \in \mathbb{Z}$, it must be the case that $a \leq b$ or $b \leq a$
- What happens if we replace \leq with $<$?

The relation $<$ is not reflexive, and $(\mathbb{Z}, <)$ is not a poset

Well Orderings: Definition

- **Definition:** (S, \preccurlyeq) is a well-ordered set if
 - It is a poset
 - Such that \preccurlyeq is a total ordering and
 - Such that every non-empty subset of S has a least element
- Example
 - The natural numbers along with \leq , (\mathbb{N}, \leq) , is a well-ordered set since any nonempty subset of \mathbb{N} has a least element and \leq is a total ordering on \mathbb{N}
 - However, (\mathbb{Z}, \leq) is not a well-ordered set
 - Why? $\mathbb{Z}^- \subset \mathbb{Z}$ but does not have a least element
 - Is it totally ordered? Yes

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- Topological Sorting

Principle of Well-Ordered Induction

- Well-ordered sets are the basis of the proof technique known as induction (more when we cover Chapter 3)
- **Theorem: Principle of Well-Ordered Induction**

Given S is a well-ordered set. $P(x)$ is true for all $x \in S$ if

(**Basis Step:** $P(x_0)$ is true for the least element in S and)

Inductive Step: For every $y \in S$ if $P(x)$ is true for all $x \prec y$, then $P(y)$ is true

Principle of Well-Ordered Induction: Proof

Proof: $(S \text{ well ordered}) \wedge (\text{Basis Step}) \wedge (\text{Induction Step}) \Rightarrow \forall x \in S, P(x)$

- Suppose that it is not the case the $P(x)$ holds for all $x \in S$
 - $\Rightarrow \exists y P(y)$ is false
 - $\Rightarrow A = \{ x \in S \mid P(x) \text{ is false} \}$ is not empty
- S is well ordered $\Rightarrow A$ has a least element a
- Since $P(x_0)$ is true and $P(a)$ is false $\Rightarrow a \neq x_0$
- $P(x)$ holds for all $x \in S$ and $x \prec a$, then $P(a)$ holds by the induction step
- This yields a contradiction **QED**

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- **Lexicographic orderings**
 - **Idea, on $A_1 \times A_2, A_1 \times A_2 \times \dots \times A_n, S^t$ (strings)**
- Hasse Diagrams
- Extremal elements
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Lexicographic Orderings: Idea

- Lexicographic ordering is the same as any dictionary or phone-book ordering:
 - We use alphabetic ordering
 - Starting with the first character in the string
 - Then the next character, if the first was equal, etc.
 - If a word is shorter than the other, than we consider that the ‘no character’ of the shorter word to be less than ‘a’

Lexicographic Orderings on $A_1 \times A_2$

- Formally, lexicographic ordering is defined by two other orderings
- **Definition:** Let (A_1, \preceq_1) and (A_2, \preceq_2) be two posets. The lexicographic ordering \prec on the Cartesian product $A_1 \times A_2$ is defined by
$$(a_1, a_2) \prec (a'_1, a'_2) \text{ if } (a_1 \prec_1 a'_1) \text{ or } (a_1 = a'_1 \text{ and } a_2 \prec_2 a'_2)$$
- If we add equality to the lexicographic ordering \prec on $A_1 \times A_2$, we obtain a partial ordering

Lexicographic Ordering on $A_1 \times A_2 \times \dots \times A_n$

- Lexicographic ordering generalizes to the Cartesian Product of n set in a natural way
- Define \preceq on $A_1 \times A_2 \times \dots \times A_n$ by

$$(a_1, a_2, \dots, a_n) \preceq (b_1, b_2, \dots, b_n)$$

If $a_1 \prec b_1$ or if there is an integer $i > 0$ such that

$$a_1 = b_1, a_2 = b_2, \dots, a_i = b_i \text{ and } a_{i+1} \prec b_{i+1}$$

Lexicographic Ordering on Strings

- Consider the two non-equal strings $a_1a_2\dots a_m$ and $b_1b_2\dots b_n$ on a poset (S^t, \preceq)
- Let
 - $t = \min(n, m)$
 - \prec be the lexicographic ordering on S^t
- $a_1a_2\dots a_m$ is less than $b_1b_2\dots b_n$ if and only if
 - $(a_1, a_2, \dots, a_t) \prec (b_1, b_2, \dots, b_t)$ or
 - $(a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t)$ and $m < n$

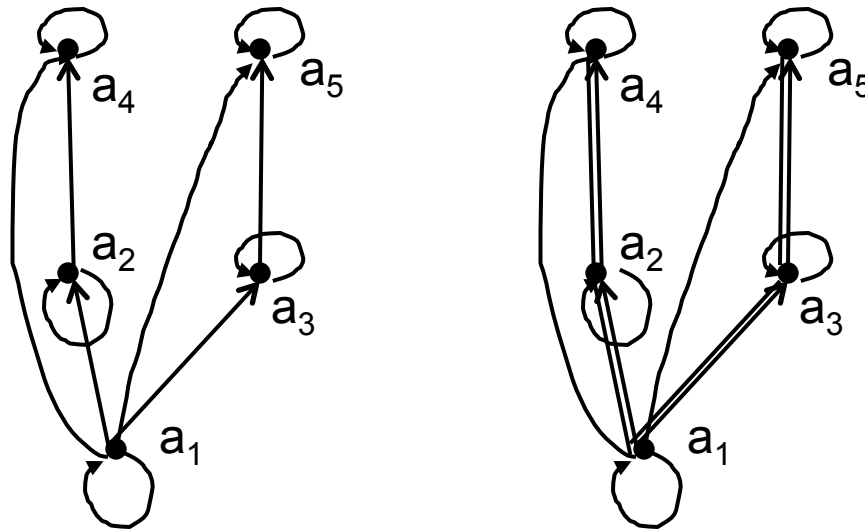
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Hasse Diagrams

- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
 - Consider the digraph representation of a partial order
 - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
 - Thus, we can simplify the graph as follows
 - Remove all self loops
 - Remove all transitive edges
 - Remove directions on edges assuming that they are oriented upwards
 - The resulting diagram is far simpler

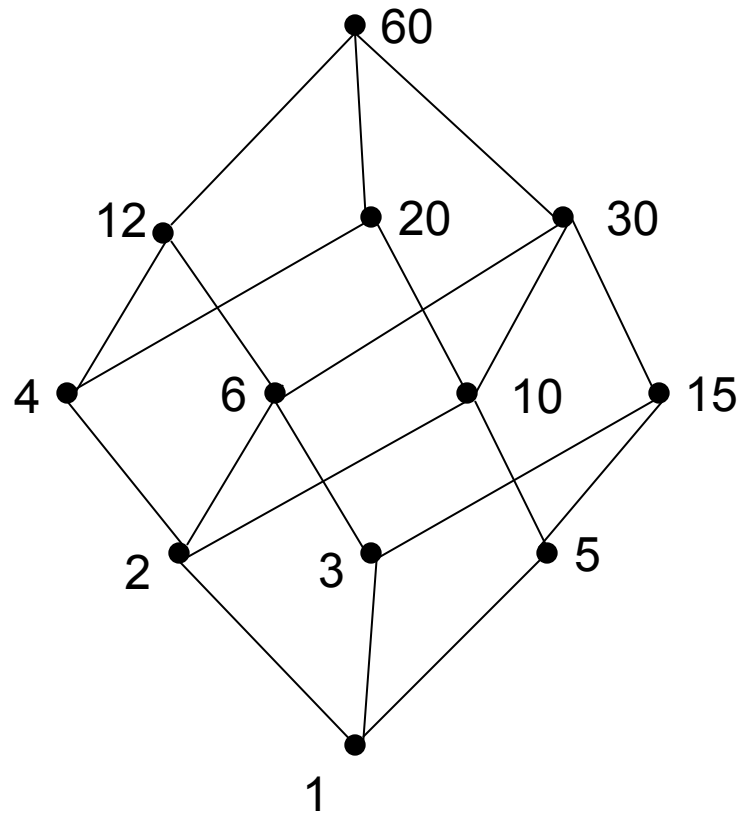
Hasse Diagram: Example



Hasse Diagrams: Example (1)

- Of course, you need not always start with the complete relation in the partial order and then trim everything.
- Rather, you can build a Hasse Diagram directly from the partial order
- Example: Draw the Hasse Diagram
 - for the following partial ordering: $\{(a,b) \mid a \mid b\}$
 - on the set $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
 - (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

Hasse Diagram: Example (2)



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Extremal Elements: Summary

We will define the following terms:

- A maximal/minimal element in a poset (S, \preceq)
- The maximum (greatest)/minimum (least) element of a poset (S, \preceq)
- An upper/lower bound element of a subset A of a poset (S, \preceq)
- The greatest lower/least upper bound element of a subset A of a poset (S, \preceq)

Extremal Elements: Maximal

- **Definition:** An element a in a poset (S, \preceq) is called maximal if it is not less than any other element in S . That is: $\neg(\exists b \in S (a \prec b))$
- If there is one unique maximal element a , we call it the maximum element (or the greatest element)

Extremal Elements: Minimal

- **Definition:** An element a in a poset (S, \preceq) is called minimal if it is not greater than any other element in S . That is: $\neg(\exists b \in S (b \prec a))$
- If there is one unique minimal element a , we call it the minimum element (or the least element)

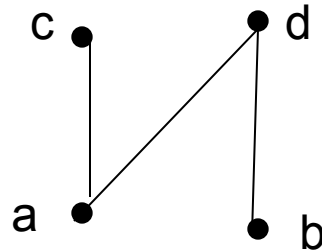
Extremal Elements: Upper Bound

- **Definition:** Let (S, \preceq) be a poset and let $A \subseteq S$. If u is an element of S such that $a \preceq u$ for all $a \in A$ then u is an upper bound of A
- An element x that is an upper bound on a subset A and is less than all other upper bounds on A is called the least upper bound on A . We abbreviate it as lub.

Extremal Elements: Lower Bound

- **Definition:** Let (S, \preceq) be a poset and let $A \subseteq S$. If l is an element of S such that $l \preceq a$ for all $a \in A$ then l is an lower bound of A
- An element x that is a lower bound on a subset A and is greater than all other lower bounds on A is called the greatest lower bound on A . We abbreviate it glb .

Extremal Elements: Example 1



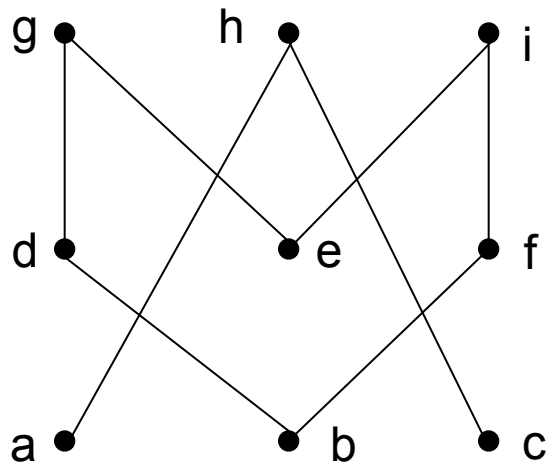
What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

Extremal Elements: Example 2

Give lower/upper bounds
& glb/lub of the sets:

$\{d,e,f\}$, $\{a,c\}$ and $\{b,d\}$



$\{d,e,f\}$

- Lower bounds: \emptyset , thus no glb
- Upper bounds: \emptyset , thus no lub

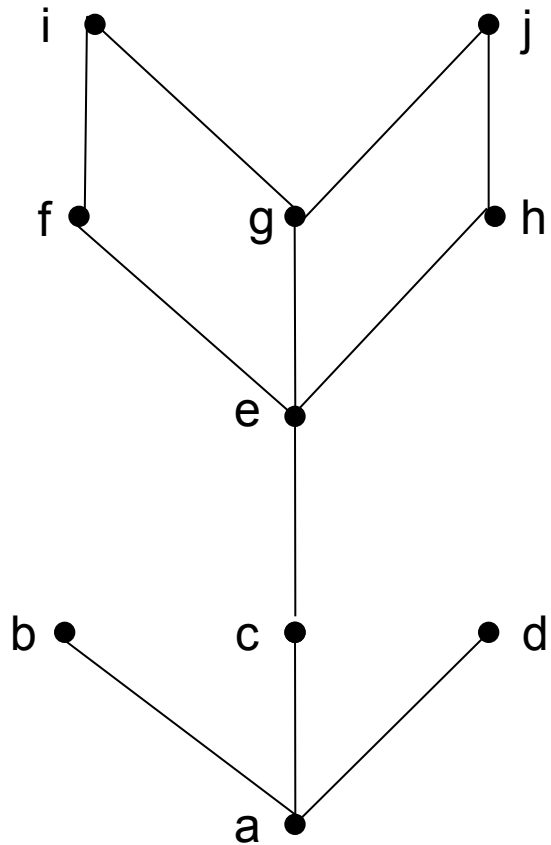
$\{a,c\}$

- Lower bounds: \emptyset , thus no glb
- Upper bounds: $\{h\}$, lub: h

$\{b,d\}$

- Lower bounds: $\{b\}$, glb: b
- Upper bounds: $\{d,g\}$, lub: d because $d < g$

Extremal Elements: Example 3



- Minimal/Maximal elements?
 - Minimal & Minimum element: a
 - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
 - Lower bounds: {a,c}, thus glb is c
 - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
 - Lower bounds: {a}, thus glb is a
 - Upper bounds: \emptyset , thus lub DNE

Outline

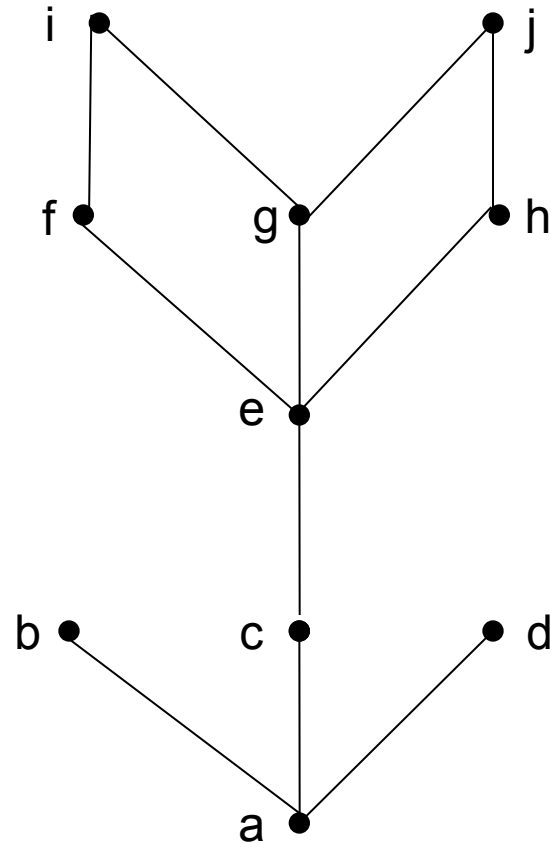
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Lattices

- A special structure arises when every pair of elements in a poset has an lub and a glb
- **Definition:** A lattice is a partially ordered set in which every pair of elements has both
 - a least upper bound and
 - a greatest lower bound

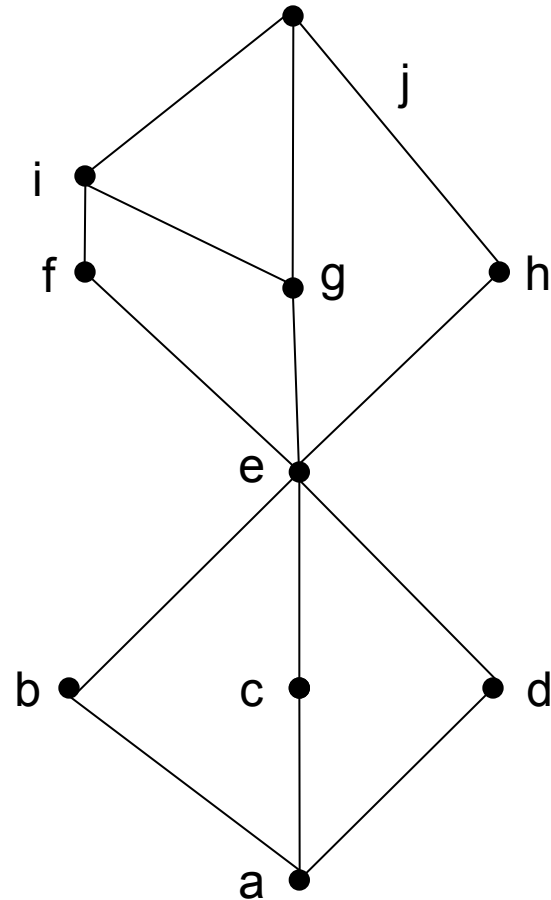
Lattices: Example 1

- Is the example from before a lattice?
- **No, because the pair $\{b,c\}$ does not have a least upper bound**



Lattices: Example 2

- What if we modified it as shown here?
- **Yes, because for any pair, there is an lub & a glb**



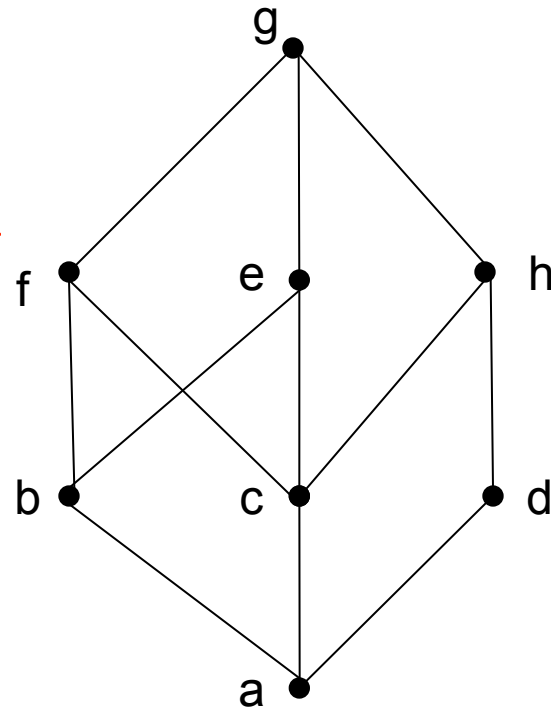
Lattices: Example 3

- Is this example a lattice?

No!

- The lower bound of $A=\{e,f\}$ is $\{a,b,c\}$
- However, A has no glb

Similarly, $B=\{b,c\}$ has no ulb



A Lattice Or Not a Lattice?

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be incomparable (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice

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Topological Sorting

- Let us return to the introductory example of Avery Hall renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule
- That is, given a partial order, we would like to transform it into a total order that is compatible with the partial order
- A total order is compatible if it does not violate any of the original relations in the partial order
- Essentially, we are simply imposing an order on incomparable elements in the partial order

Topological Sorting: Preliminaries (1)

- Before we give the algorithm, we need some tools to justify its correctness
- **Fact:** Every finite, nonempty poset (S, \preceq) has a minimal element
- We will prove the above fact by a form of *reductio ad absurdum*

Topological Sorting: Preliminaries (2)

- **Proof:**

- Assume, to the contrary, that a nonempty finite poset (S, \prec) has no minimal element. In particular, assume that a_1 is not a minimal element.
- Assume, w/o loss of generality, that $|S|=n$
- If a_1 is not minimal, then there exists a_2 such that $a_2 \prec a_1$
- But a_2 is also not minimal because of the above assumption
- Therefore, there exists a_3 such that $a_3 \prec a_2$. This process proceeds until we have the last element a_n . Thus, $a_n \prec a_{n-1} \prec \dots \prec a_2 \prec a_1$
- Finally, by definition a_n is the minimal element **QED**

Topological Sorting: Intuition

- The idea of topological sorting is
 - We start with a poset (S, \prec)
 - We remove a minimal element, choosing arbitrarily if there is more than one. Such an element is guaranteed to exist by the previous fact
 - As we remove each minimal element, one at a time, the set S shrinks
 - Thus we are guaranteed that the algorithm will terminate in a finite number of steps
 - Furthermore, the order in which the elements are removed is a total order: $a_1 \prec a_2 \prec \dots \prec a_{n-1} \prec a_n$
- Now, we can give the algorithm itself

Topological Sorting: Algorithm

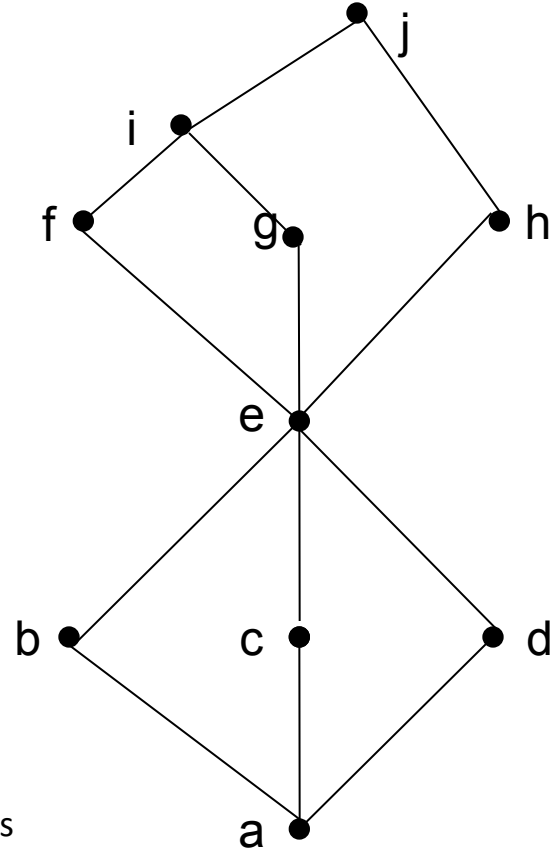
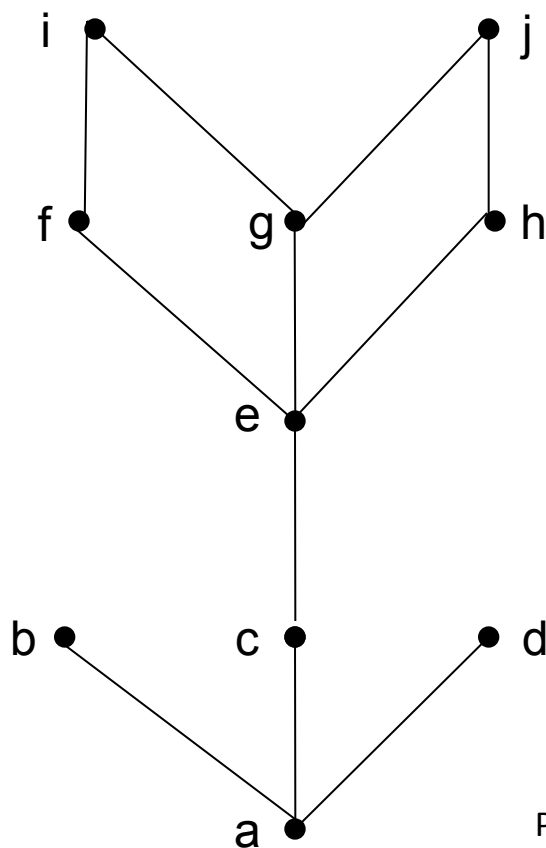
Input: (S, \preceq) a poset with $|S|=n$

Output: A total ordering (a_1, a_2, \dots, a_n)

1. $k \leftarrow 1$
2. **While** S **Do**
3. $a_k \leftarrow$ a minimal element in S
4. $S \leftarrow S \setminus \{a_k\}$
5. $k \leftarrow k+1$
6. **End**
7. **Return** (a_1, a_2, \dots, a_n)

Topological Sorting: Example

- Find a compatible ordering (topological ordering) of the poset represented by the Hasse diagrams below



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- Hasse Diagrams
- Extremal elements
 - Minimal/minimum, maximal/maximum, glb, lub
- Lattices
- Topological Sorting