

# Asymptotics

## Section 3.2 of Rosen

Spring 2013

CSCE 235 Introduction to Discrete Structures

Course web-page: [cse.unl.edu/~cse235](http://cse.unl.edu/~cse235)

**Questions:** Piazza

# Outline

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- Introduction
- Asymptotic
  - Definitions (Big O, Omega, Theta), properties
- Proof techniques
  - 3 examples, trick for polynomials of degree 2,
  - Limit method (l' Hôpital Rule), 2 examples
- Limit Properties
- Efficiency classes
- Conclusions

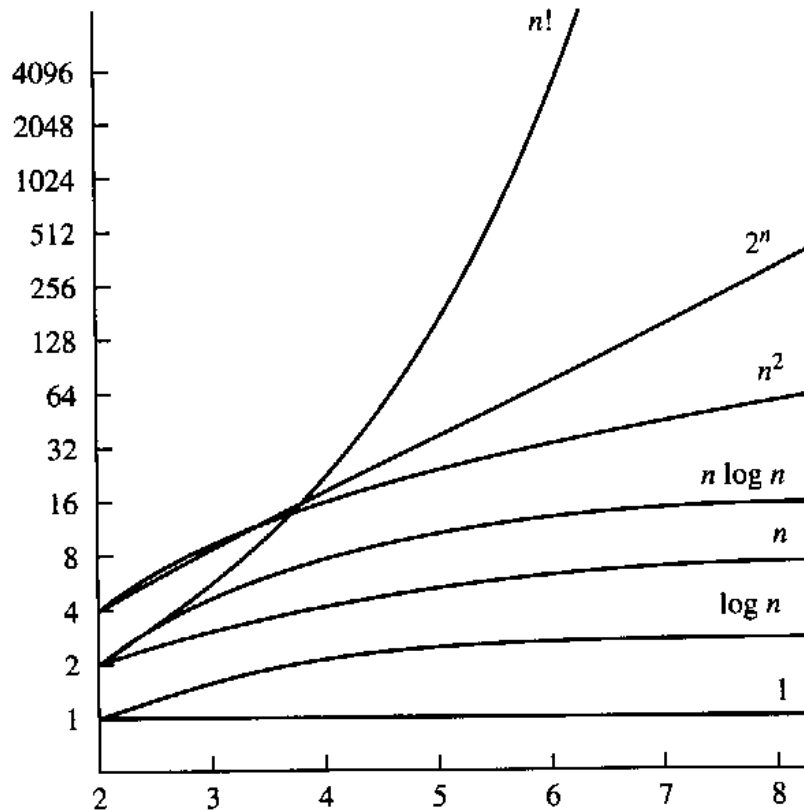
# Introduction (1)

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- We are interested only in the Order of Growth of an algorithm's complexity
- How well does the algorithm perform as the size of the input grows:  $n \rightarrow \infty$
- We have seen how to mathematically evaluate the cost functions of algorithms with respect to
  - their input size  $n$  and
  - their elementary operations
- However, it suffices to simply measure a cost function's asymptotic behavior

# Introduction (2): Magnitude Graph

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**FIGURE 3** A Display of the Growth of Functions Commonly Used in Big- $O$  Estimates.

# Introduction (3)

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- In practice, specific hardware, implementation, languages, etc. greatly affect how the algorithm behave
- Our goal is to study and analyze the behavior of algorithms in and of themselves, independently of such factors
- For example
  - An algorithm that executes its elementary operation  $10n$  times is better than one that executes it  $0.005n^2$  times
  - Also, algorithms that have running time  $n^2$  and  $2000n^2$  are considered asymptotically equivalent

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- **Asymptotic**
  - **Definitions (Big-O, Omega, Theta), properties**
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# Big-O Definition

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- **Definition:** Let  $f$  and  $g$  be two functions  $f, g: N \rightarrow R^+$ . We say that

$$f(n) \in O(g(n))$$

(read:  $f$  is Big-O of  $g$ ) if there exists a constant  $c \in R^+$  and an  $n_0 \in N$  such that for every integer  $n \geq n_0$  we have

$$f(n) \leq cg(n)$$

- Big-O is actually Omicron, but it suffices to write “O”
- Intuition:  $f$  is asymptotically less than or equal to  $g$
- Big-O gives an asymptotic upper bound

# Big-Omega Definition

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- **Definition:** Let  $f$  and  $g$  be two functions  $f, g: N \rightarrow R^+$ . We say that

$$f(n) \in \Omega(g(n))$$

(read:  $f$  is Big-Omega of  $g$ ) if there exists a constant  $c \in R^+$  and an  $n_0 \in N$  such that for every integer  $n \geq n_0$  we have

$$f(n) \geq cg(n)$$

- Intuition:  $f$  is asymptotically greater than or equal to  $g$
- Big-Omega gives an asymptotic lower bound



# Big-Theta Definition

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- **Definition:** Let  $f$  and  $g$  be two functions  $f, g: N \rightarrow R^+$ . We say that

$$f(n) \in \Theta(g(n))$$

(read:  $f$  is Big-Omega of  $g$ ) if there exists a constant  $c_1, c_2 \in R^+$  and an  $n_0 \in N$  such that for every integer  $n \geq n_0$  we have

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

- Intuition:  $f$  is asymptotically equal to  $g$
- $f$  is bounded above and below by  $g$
- Big-Theta gives an asymptotic equivalence

# Asymptotic Properties (1)

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- **Theorem:** For  $f_1(n) \in O(g_1(n))$  and  $f_2(n) \in O(g_2(n))$ , we have
$$f_1(n) + f_2(n) \in O(\max\{g_1(n), g_2(n)\})$$

- This property implies that we can ignore lower order terms. In particular, for any polynomial with degree  $k$  such as
$$p(n) = an^k + bn^{k-1} + cn^{k-2} + \dots,$$

$$p(n) \in O(n^k)$$

*More accurately,  $p(n) \in \Theta(n^k)$*

- In addition, this theorem gives us a justification for ignoring constant coefficients. That is for any function  $f(n)$  and a **positive** constant  $c$

$$cf(n) \in \Theta(f(n))$$

# Asymptotic Properties (2)

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- Some obvious properties also follow from the definitions
  - **Corollary:** For positive functions  $f(n)$  and  $g(n)$  the following hold:
    - $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n)) \wedge f(n) \in \Omega(g(n))$
    - $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- The proof is obvious and left as an exercise

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# Asymptotic Proof Techniques

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- Proving an asymptotic relationship between two given function  $f(n)$  and  $g(n)$  can be done intuitively for most of the functions you will encounter; all polynomials for example
- However, this does not suffice as a formal proof
- To prove a relationship of the form  $f(n) \in \Delta(g(n))$ , where  $\Delta$  is  $O$ ,  $\Omega$ , or  $\Theta$ , can be done using the definitions, that is
  - Find a value for  $c$  (or  $c_1$  and  $c_2$ )
  - Find a value for  $n_0$(But the above is not the only way.)

# Asymptotic Proof Techniques: Example A

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**Example:** Let  $f(n)=21n^2+n$  and  $g(n)=n^3$

- Our intuition should tell us that  $f(n) \in O(g(n))$
- Simply using the definition confirms this:

$$21n^2+n \leq cn^3$$

holds for **say**  $c=3$  and for all  $n \geq n_0=8$

- So we found a pair  $c=3$  and  $n_0=8$  that satisfy the conditions required by the definition

**QED**

- In fact, an infinite number of pairs can satisfy this equation

# Asymptotic Proof Techniques: Example B (1)

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- **Example:** Let  $f(n)=n^2+n$  and  $g(n)=n^3$ . Find a tight bound of the form

$$f(n) \in \Delta(g(n))$$

- Our intuition tells us that  $f(n) \in O(g(n))$
- Let's prove it formally

# Example B: Proof

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- If  $n \geq 1$  it is clear that

1.  $n \leq n^3$  and

2.  $n^2 \leq n^3$

- Therefore, we have, as 1. and 2.:

$$n^2 + n \leq n^3 + n^3 = 2n^3$$

- Thus, for  $n_0=1$  and  $c=2$ , by the definition of Big-O we have that  $f(n)=n^2+n \in O(g(n^3))$



# Asymptotic Proof Techniques: Example C (1)

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- **Example:** Let  $f(n)=n^3+4n^2$  and  $g(n)=n^2$ . Find a tight bound of the form

$$f(n) \in \Delta(g(n))$$

- Here, Our intuition tells us that  $f(n) \in \Omega(g(n))$
- Let's prove it formally

# Example C: Proof

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- For  $n \geq 1$ , we have  $n^2 \leq n^3$
- For  $n \geq 0$ , we have  $n^3 \leq n^3 + 4n^2$
- Thus  $n \geq 1$ , we have  $n^2 \leq n^3 \leq n^3 + 4n^2$
- Thus, by the definition of Big- $\Omega$  , for  $n_0=1$  and  $c=1$  we have that  $f(n)=n^3+4n^2 \in \Omega(g(n^2))$

# Asymptotic Proof Techniques:

## Trick for polynomials of degree 2

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- If you have a polynomial of degree 2 such as

$$an^2+bn+c$$

you can prove that it is  $\Theta(n^2)$  using the following values

1.  $c_1=a/4$
2.  $c_2=7a/4$
3.  $n_0= 2 \max(|b|/a, \text{sqrt}(|c|)/a)$

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# Limit Method: Motivation

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- Now try this one:

$$f(n) = n^{50} + 12n^3 \log^4 n - 1243n^{12}$$

$$+ 245n^6 \log n + 12 \log^3 n - \log n$$

$$g(n) = 12 n^{50} + 24 \log^{14} n^{43} - \log n / n^5 + 12$$

- Using the formal definitions can be very tedious especially one has very complex functions
- It is much better to use the Limit Method, which uses concepts from Calculus

# Limit Method: The Process

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- Say we have functions  $f(n)$  and  $g(n)$ . We set up a limit quotient between  $f$  and  $g$  as follows

$$\lim_{n \rightarrow \infty} f(n)/g(n) = \begin{cases} 0 & \text{Then } f(n) \in O(g(n)) \\ c > 0 & \text{Then } f(n) \in \Theta(g(n)) \\ \infty & \text{Then } f(n) \in \Omega(g(n)) \end{cases}$$

- The above can be proven using calculus, but for our purposes, the limit method is sufficient for showing asymptotic inclusions
- Always try to look for algebraic simplifications first
- If  $f$  and  $g$  both diverge or converge on zero or infinity, then you need to apply the l' Hôpital Rule

# (Guillaume de) L' Hôpital Rule

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- Theorem (L' Hôpital Rule):
  - Let  $f$  and  $g$  be two functions,
  - if the limit between the quotient  $f(n)/g(n)$  exists,
  - Then, it is equal to the limit of the derivative of the numerator and the denominator

$$\lim_{n \rightarrow \infty} f(n)/g(n) = \lim_{n \rightarrow \infty} f'(n)/g'(n)$$

# Useful Identities & Derivatives

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- Some useful derivatives that you should memorize
  - $(n^k)' = k n^{k-1}$
  - $(\log_b(n))' = 1/(n \ln(b))$
  - $(f_1(n)f_2(n))' = f_1'(n)f_2(n) + f_1(n)f_2'(n)$  (*product rule*)
  - $(\log_b(f(n)))' = f'(n)/(f(n) \ln b)$
  - $(c^n)' = \ln(c)c^n$  ← *careful!*
- Log identities
  - Change of base formula:  $\log_b(n) = \log_c(n)/\log_c(b)$
  - $\log(n^k) = k \log(n)$
  - $\log(ab) = \log(a) + \log(b)$



# L' Hôpital Rule: Justification (1)

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- Why do we have to use L' Hôpital' s Rule?
- Consider the following function

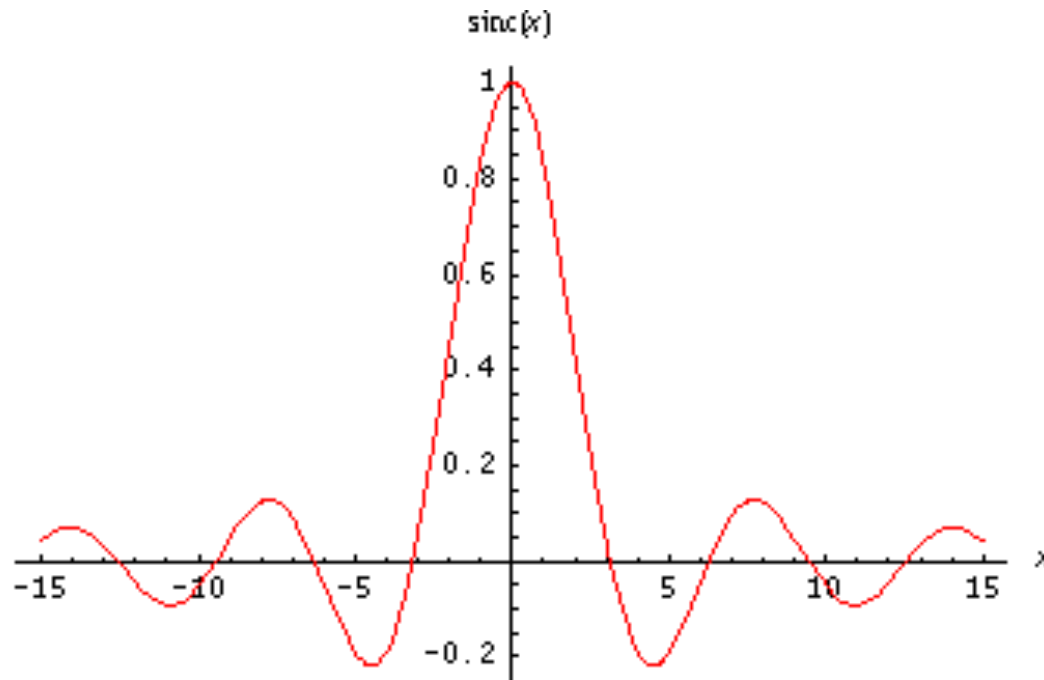
$$f(x) = (\sin x)/x$$

- Clearly  $\sin 0=0$ . So you may say that when  $x \rightarrow 0$ ,  $f(x) \rightarrow 0$
- However, the denominator is also  $\rightarrow 0$ , so you may say that  $f(x) \rightarrow \infty$
- Both are wrong

# L' Hôpital Rule: Justification (2)

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- Observe the graph of  $f(x) = (\sin x)/x = \text{sinc } x$



# L' Hôpital Rule: Justification (3)

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- Clearly, though  $f(x)$  is undefined at  $x=0$ , the limit still exists
- Applying the L' Hôpital Rule gives us the correct answer

$$\lim_{x \rightarrow 0} ((\sin x)/x) = \lim_{x \rightarrow 0} (\sin x)' / x' = \cos x / 1 = 1$$

# Limit Method: Example 1

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- Example: Let  $f(n) = 2^n$ ,  $g(n) = 3^n$ . Determine a tight inclusion of the form  $f(n) \in \Delta(g(n))$
- What is your intuition in this case? Which function grows quicker?

# Limit Method: Example 1—Proof A

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- Proof using limits
- We set up our limit:

$$\lim_{n \rightarrow \infty} f(n)/g(n) = \lim_{n \rightarrow \infty} 2^n/3^n$$

- Using L' Hôpital Rule gets you no where

$$\lim_{n \rightarrow \infty} 2^n/3^n = \lim_{n \rightarrow \infty} (2^n)' / (3^n)' = \lim_{n \rightarrow \infty} (\ln 2)(2^n) / (\ln 3)(3^n)$$

- Both the numerator and denominator still diverge. We'll have to use an algebraic simplification

# Limit Method: Example 1—Proof B

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- Using algebra

$$\lim_{n \rightarrow \infty} 2^n/3^n = \lim_{n \rightarrow \infty} (2/3)^n$$

- Now we use the following Theorem w/o proof

$$\lim_{n \rightarrow \infty} \alpha^n = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

- Therefore we conclude that the  $\lim_{n \rightarrow \infty} (2/3)^n$  converges to zero thus  $2^n \in O(3^n)$

# Limit Method: Example 2 (1)

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- Example: Let  $f(n) = \log_2 n$ ,  $g(n) = \log_3 n^2$ .  
Determine a tight inclusion of the form
$$f(n) \in \Delta(g(n))$$
- What is your intuition in this case?

# Limit Method: Example 2 (2)

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- We prove using limits
- We set up our limit

$$\begin{aligned}\lim_{n \rightarrow \infty} f(n)/g(n) &= \lim_{n \rightarrow \infty} \log_2 n / \log_3 n^2 \\ &= \lim_{n \rightarrow \infty} \log_2 n / (2 \log_3 n)\end{aligned}$$

- Here we use the change of base formula for logarithms:  $\log_x n = \log_y n / \log_y x$
- Thus:  $\log_3 n = \log_2 n / \log_2 3$



# Limit Method: Example 2 (3)

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- Computing our limit:

$$\begin{aligned}\lim_{n \rightarrow \infty} \log_2 n / (2 \log_3 n) &= \lim_{n \rightarrow \infty} \log_2 n \log_2 3 / (2 \log_2 n) \\ &= \lim_{n \rightarrow \infty} (\log_2 3) / 2 \\ &= (\log_2 3) / 2 \\ &\approx 0.7924, \text{ which is a positive constant}\end{aligned}$$

- So we conclude that  $f(n) \in \Theta(g(n))$

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# Limit Properties

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- A useful property of limits is that the composition of functions is preserved
- **Lemma:** For the composition  $\circ$  of addition, subtraction, multiplication and division, if the limits exist (that is, they converge), then

$$\lim_{n \rightarrow \infty} f_1(n) \circ \lim_{n \rightarrow \infty} f_2(n) = \lim_{n \rightarrow \infty} (f_1(n) \circ f_2(n))$$

# Efficiency Classes—Table 1, page 196

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- Constant  $O(1)$
- Logarithmic  $O(\log (n))$
- Linear  $O(n)$
- Polylogarithmic  $O(\log^k (n))$
- Quadratic  $O(n^2)$
- Cubic  $O(n^3)$
- Polynomial  $O(n^k)$  for any  $k>0$
- Exponential  $O(k^n)$ , where  $k>1$
- Factorial  $O(n!)$

# Conclusions

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- Evaluating asymptotics is easy, but remember:
  - **Always** look for algebraic simplifications
  - You must **always** give a rigorous proof
  - Using the limit method is (almost) always the best
  - Use L' Hôpital Rule if need be
  - Give as simple **and tight** expressions as possible

# Summary

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