

# Sets

**Sections 2.1 and 2.2 of Rosen**

Spring 2012

CSCE 235 Introduction to Discrete Structures

Course web-page: [cse.unl.edu/~cse235](http://cse.unl.edu/~cse235)

Questions: [cse235@cse.unl.edu](mailto:cse235@cse.unl.edu)

# Outline

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- Definitions: set, element
- Terminology and notation
  - Set equal, multi-set, bag, set builder, intension, extension, Venn Diagram (representation), empty set, singleton set, subset, proper subset, finite/infinite set, cardinality
- Proving equivalences
- Power set
- Tuples (ordered pair)
- Cartesian Product (a.k.a. Cross product), relation
- Quantifiers
- Set Operations (union, intersection, complement, difference), Disjoint sets
- Set equivalences (cheat sheet or Table 1, page 130)
  - Inclusion in both directions
  - Using membership tables
- Generalized Unions and Intersection
- Computer Representation of Sets

# Introduction (1)

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- We have already implicitly dealt with sets
  - Integers ( $\mathbb{Z}$ ), rationals ( $\mathbb{Q}$ ), naturals ( $\mathbb{N}$ ), reals ( $\mathbb{R}$ ), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition:** A set is an unordered collection of (unique) objects
- Sets are fundamental discrete structures and form the basis of more complex discrete structures like graphs

# Introduction (2)

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- **Definition:** The objects in a set are called elements or members of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$ : x is an element of A  $\in$
  - $x \notin A$ : x is not an element of A  $\notin$

# Terminology (1)

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- **Definition:** Two sets, A and B, are equal if they contain the same elements. We write  $A=B$ .
- Example:
  - $\{2,3,5,7\}=\{3,2,7,5\}$ , because a set is unordered
  - Also,  $\{2,3,5,7\}=\{2,2,3,5,3,7\}$  because a set contains unique elements
  - However,  $\{2,3,5,7\} \neq \{2,3\}$   $\neq$

# Terminology (2)

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- A multi-set is a set where you specify the number of occurrences of each element:  $\{m_1 \cdot a_1, m_2 \cdot a_2, \dots, m_r \cdot a_r\}$  is a set where
  - $m_1$  occurs  $a_1$  times
  - $m_2$  occurs  $a_2$  times
  - ...
  - $m_r$  occurs  $a_r$  times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

# Terminology (3)

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- The **set-builder** notation

$$O = \{ x \mid (x \in \mathbb{Z}) \wedge (x = 2k) \text{ for some } k \in \mathbb{Z} \}$$

reads:  $O$  is the set that contains all  $x$  such that  $x$  is an integer and  $x$  is even

- A set is defined in **intension** when you give its set-builder notation

$$O = \{ x \mid (x \in \mathbb{Z}) \wedge (0 \leq x \leq 8) \wedge (x = 2k) \text{ for some } k \in \mathbb{Z} \}$$

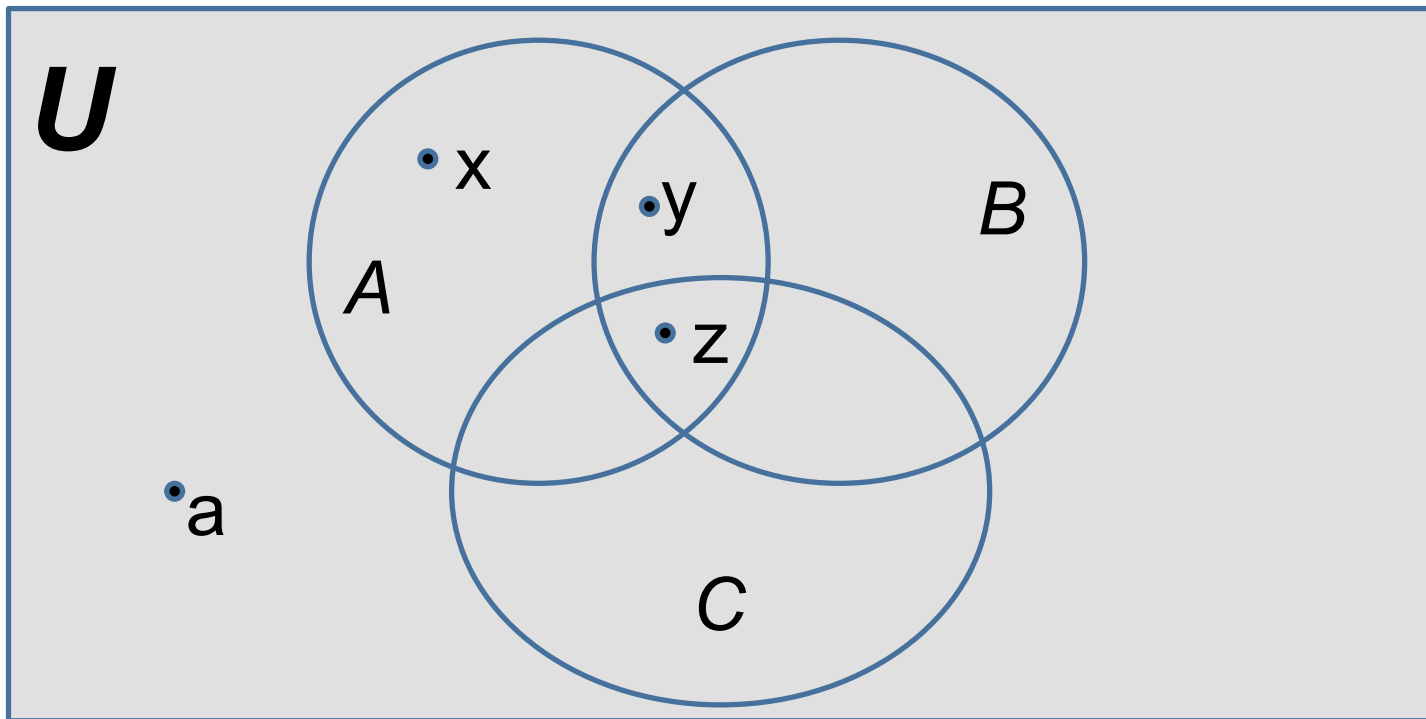
- A set is defined in **extension** when you enumerate all the elements:

$$O = \{0, 2, 4, 6, 8\}$$

# Venn Diagram: Example

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- A set can be represented graphically using a Venn Diagram





# More Terminology and Notation (1)

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- A set that has no elements is called the **empty set** or **null set** and is denoted  $\emptyset$  `\emptyset`
- A set that has one element is called a **singleton set**.
  - For example:  $\{a\}$ , with brackets, is a singleton set
  - $a$ , without brackets, is an element of the set  $\{a\}$
- Note the subtlety in  $\emptyset \neq \{\emptyset\}$ 
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

# More Terminology and Notation (2)

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- **Definition:** A is said to be a **subset** of B, and we write  $A \subseteq B$ , if and only if every element of A is also an element of B `\subseteq`

- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x (x \in A \Rightarrow x \in B)$$

# More Terminology and Notation (3)

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- **Theorem:** For any set  $S$  *Theorem 1, page 120*
  - $\emptyset \subseteq S$  and
  - $S \subseteq S$
- The proof is in the book, an excellent example of a vacuous proof

# More Terminology and Notation (4)

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- **Definition:** A set  $A$  that is a subset of a set  $B$  is called a **proper subset** if  $A \neq B$ .
- That is there is an element  $x \in B$  such that  $x \notin A$
- We write:  $A \subset B$ ,  $A \subsetneq B$
- In LaTeX:  $\setminus subset$ ,  $\setminus subsetneq$

# More Terminology and Notation (5)

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- Sets can be elements of other sets
- Examples
  - $S_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, c\}$
  - $S_2 = \{\{1\}, \{2, 4, 8\}, \{3\}, \{6\}, 4, 5, 6\}$

# More Terminology and Notation (6)

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- **Definition:** If there are exactly  $n$  distinct elements in a set  $S$ , with  $n$  a nonnegative integer, we say that:
  - $S$  is a **finite set**, and
  - The **cardinality** of  $S$  is  $n$ . Notation:  $|S| = n$ .
- **Definition:** A set that is not finite is said to be **infinite**

# More Terminology and Notation (7)

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- Examples
  - Let  $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$ , the cardinality of  $B$  is  $|B|=25$  because there are 25 primes less than or equal to 100.
  - The cardinality of the empty set is  $|\emptyset|=0$
  - The sets  $N, Z, Q, R$  are all infinite

# Proving Equivalence (1)

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- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a **subset** of B, use the equivalence discussed earlier  $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$ 
  - To prove that  $A \subseteq B$  it is enough to show that for an arbitrary (nonspecific) element  $x$ ,  $x \in A$  implies that  $x$  is also in B.
  - Any proof method can be used.
- To prove that A is a **proper subset** of B, you must prove
  - A is a subset of B **and**
  - $\exists x (x \in B) \wedge (x \notin A)$



# Proving Equivalence (2)

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- Finally to show that two sets are **equal**, it is sufficient to show independently (much like a biconditional) that

- $A \subseteq B$  and

- $B \subseteq A$

- Logically speaking, you must show the following quantified statements:

$$(\forall x (x \in A \Rightarrow x \in B)) \wedge (\forall x (x \in B \Rightarrow x \in A))$$

we will see an example later..

# Power Set (1)

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- **Definition:** The power set of a set  $S$ , denoted  $P(S)$ , is the set of all subsets of  $S$ .
- Examples
  - Let  $A=\{a,b,c\}$ ,  $P(A)=\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$
  - Let  $A=\{\{a,b\}, c\}$ ,  $P(A)=\{\emptyset, \{\{a,b\}\}, \{c\}, \{\{a,b\}, c\}\}$
- Note: the empty set  $\emptyset$  and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 120).

# Power Set (2)

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- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact:** Let  $S$  be a set such that  $|S|=n$ , then

$$|P(S)| = 2^n$$

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- Power set
- **Tuples (ordered pair)**
- **Cartesian Product (a.k.a. Cross product), relation**
- **Quantifiers**
- Set Operations (union, intersection, complement, difference), Disjoint sets
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# Tuples (1)

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- Sometimes we need to consider **ordered** collections of objects
- **Definition:** The ordered  $n$ -tuple  $(a_1, a_2, \dots, a_n)$  is the ordered collection with the element  $a_i$  being the  $i$ -th element for  $i=1, 2, \dots, n$
- Two ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$  are equal iff for every  $i=1, 2, \dots, n$  we have  $a_i = b_i$
- A 2-tuple ( $n=2$ ) is called an **ordered pair**

# Cartesian Product (1)

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- **Definition:** Let  $A$  and  $B$  be two sets. The **Cartesian product** of  $A$  and  $B$ , denoted  $A \times B$ , is the set of all ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$

$$A \times B = \{ (a, b) \mid (a \in A) \wedge (b \in B) \}$$

- The Cartesian product is also known as the **cross product**
- **Definition:** A subset of a Cartesian product,  $R \subseteq A \times B$  is called a **relation**. We will talk more about relations in the next set of slides
- Note:  $A \times B \neq B \times A$  unless  $A = \emptyset$  or  $B = \emptyset$  or  $A = B$ . Find a counter example to prove this.

# Cartesian Product (2)

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- Cartesian Products can be generalized for any n-tuple
- **Definition:** The Cartesian product of n sets,  $A_1, A_2, \dots, A_n$ , denoted  $A_1 \times A_2 \times \dots \times A_n$ , is
$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \dots, n \}$$

# Notation with Quantifiers

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- Whenever we wrote  $\exists xP(x)$  or  $\forall xP(x)$ , we specified the universe of discourse using explicit English language
- Now we can simplify things using set notation!
- Example
  - $\forall x \in R (x^2 \geq 0)$
  - $\exists x \in Z (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a, b, c \in R \exists x \in C (ax^2 + bx + c = 0)$$



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# Set Operations

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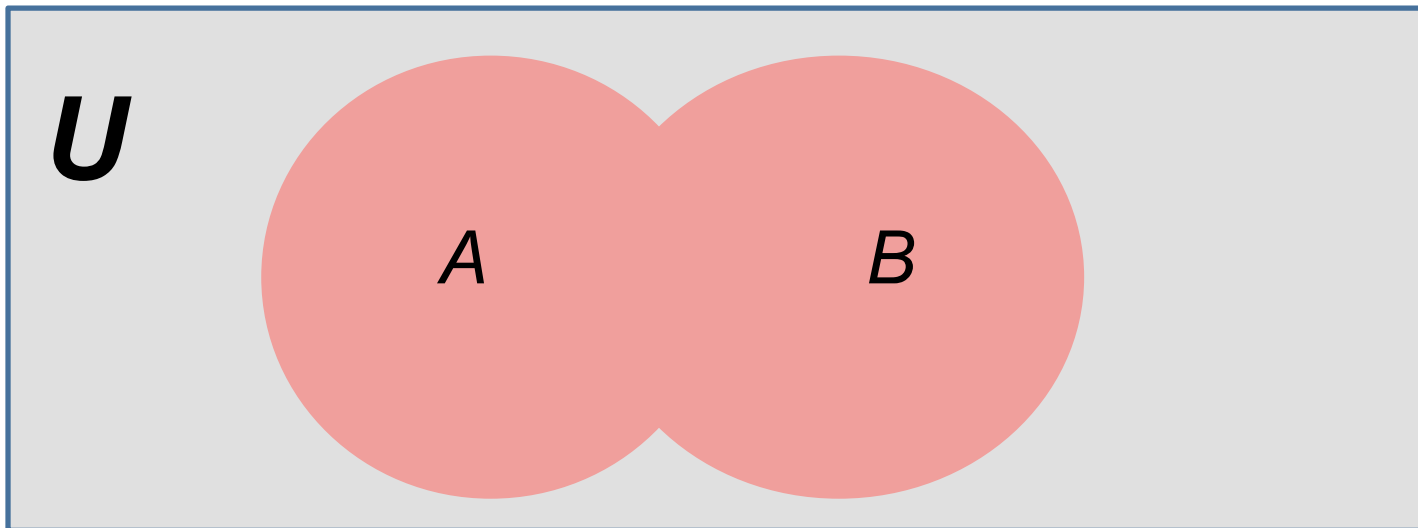
- Arithmetic operators (+, -,  $\times$ ,  $\div$ ) can be used on pairs of numbers to give us new numbers
- Similarly, set operators exist and act on two sets to give us new sets
  - Union  $\cup$
  - Intersection  $\cap$
  - Set difference  $\setminus$
  - Set complement  $\overline{S}$
  - Generalized union  $\bigcup$
  - Generalized intersection  $\bigcap$

# Set Operators: Union

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- **Definition:** The **union** of two sets A and B is the set that contains all elements in A, B, or both. We write:

$$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$$

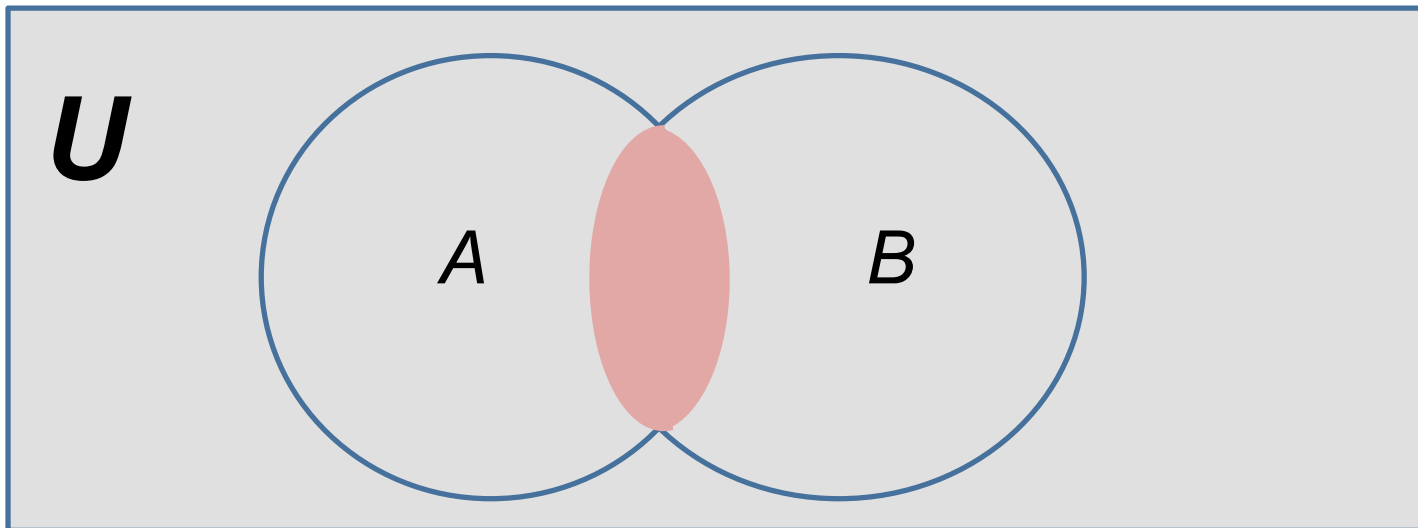


# Set Operators: Intersection

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- **Definition:** The **intersection** of two sets  $A$  and  $B$  is the set that contains all elements that are element of both  $A$  and  $B$ . We write:

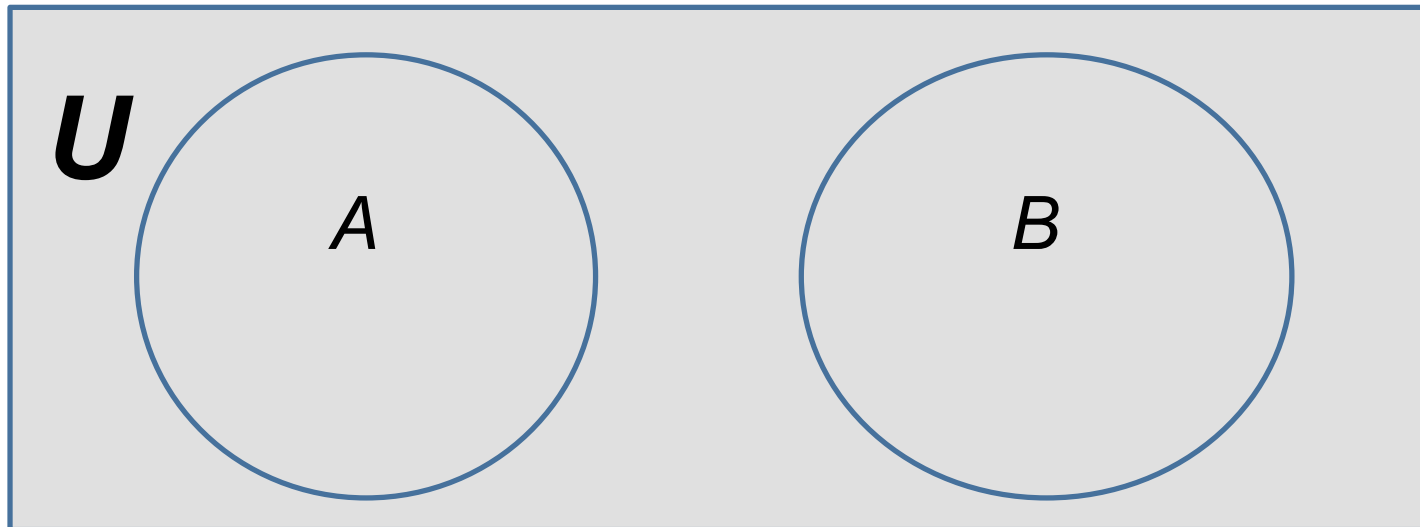
$$A \cap B = \{ x \mid (x \in A) \wedge (x \in B) \}$$



# Disjoint Sets

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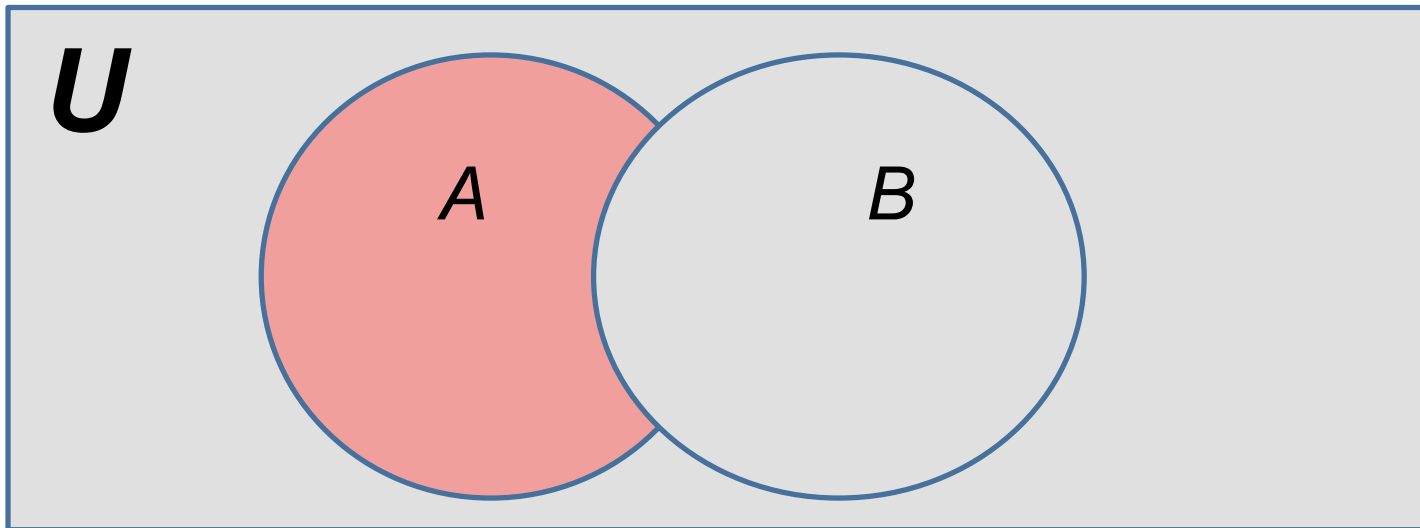
- **Definition:** Two sets are said to be **disjoint** if their intersection is the empty set:  $A \cap B = \emptyset$



# Set Difference

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- **Definition:** The **difference** of two sets  $A$  and  $B$ , denoted  $A \setminus B$  ( $\setminus$  setminus) or  $A - B$ , is the set containing those elements that are in  $A$  but not in  $B$

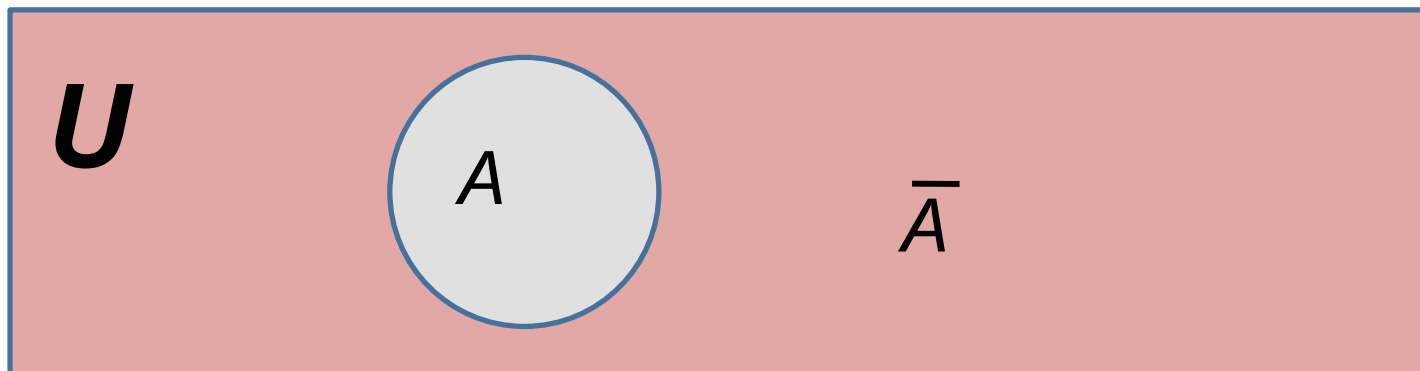


# Set Complement

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- **Definition:** The **complement** of a set  $A$ , denoted  $\bar{A}$  ( $\bar{\phantom{A}}$ ), consists of all elements not in  $A$ . That is the difference of the universal set and  $U$ :  $U \setminus A$

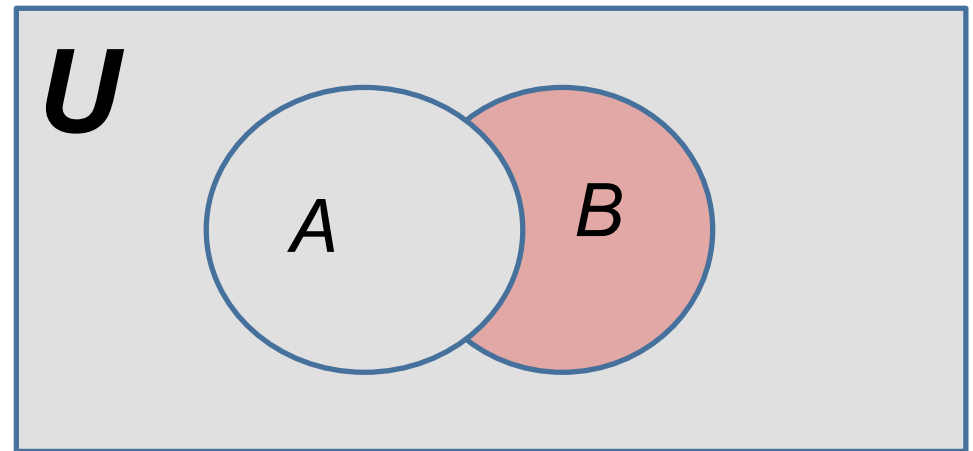
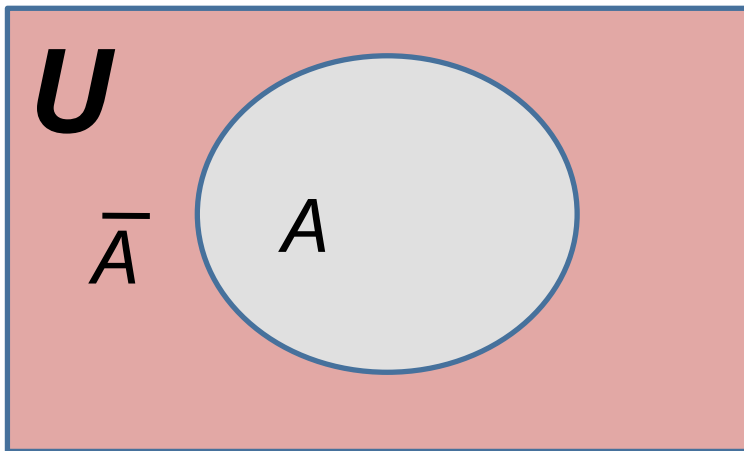
$$\bar{A} = A^c = \{x \mid x \notin A\}$$



# Set Complement: Absolute & Relative

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- Given the Universe  $U$ , and  $A, B \subset U$ .
- The (absolute) complement of  $A$  is  $A^c = U \setminus A$
- The (relative) complement of  $A$  in  $B$  is  $B \setminus A$





# Set Identities

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- There are analogs of all the usual laws for set operations. Again, the Cheat Sheet is available on the course webpage:  
<http://www.cse.unl.edu/~cse235/files/LogicalEquivalences.pdf>
- Let's take a quick look at this Cheat Sheet or at Table 1 on page 130 in your textbook

# Proving Set Equivalences

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- Recall that to prove such identity, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal
- The book proves several of the standard set identities
- We will give a couple of different examples here

# Proving Set Equivalences: Example A (1)

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- Let
  - $A = \{x \mid x \text{ is even}\}$
  - $B = \{x \mid x \text{ is a multiple of } 3\}$
  - $C = \{x \mid x \text{ is a multiple of } 6\}$
- Show that  $A \cap B = C$

## Proving Set Equivalences: Example A (2)

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- **$A \cap B \subseteq C$ :  $\forall x \in A \cap B$** 
  - $\Rightarrow x$  is a multiple of 2 and  $x$  is a multiple of 3
  - $\Rightarrow$  we can write  $x=2 \cdot 3 \cdot k$  for some integer  $k$
  - $\Rightarrow x=6k$  for some integer  $k \Rightarrow x$  is a multiple of 6
  - $\Rightarrow x \in C$
- **$C \subseteq A \cap B$ :  $\forall x \in C$** 
  - $\Rightarrow x$  is a multiple of 6  $\Rightarrow x=6k$  for some integer  $k$
  - $\Rightarrow x=2(3k)=3(2k) \Rightarrow x$  is a multiple of 2 and of 3
  - $\Rightarrow x \in A \cap B$

# Proving Set Equivalences: Example B (1)

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- An alternative prove is to use **membership tables** where an entry is
  - 1 if a chosen (but fixed) element is in the set
  - 0 otherwise
- Example: Show that

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

## Proving Set Equivalences: Example B (2)

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A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A \cup B \cup C}$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	0	1	1
0	1	0	0	1	0	1	1	1
0	1	1	0	1	0	0	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0

- 1 under a set indicates that “an element is in the set”
- If the columns are equivalent, we can conclude that indeed the two sets are equal

## Generalizing Set Operations: Union and Intersection

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- In the previous example, we showed De Morgan's Law generalized to unions involving 3 sets
- In fact, De Morgan's Laws hold for any finite set of sets
- Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets

# Generalized Union

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- **Definition:** The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

LaTeX:  $\$\Bigcup_{i=1}^n A_i=A_1\cup A_2 \cup\ldots\cup A_n\$$



# Generalized Intersection

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- **Definition:** The **intersection of a collection of sets** is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

LaTeX:  $\$\Bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n\$$

# Computer Representation of Sets (1)

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- There really aren't ways to represent infinite sets by a computer since a computer has a finite amount of memory
- If we assume that the universal set  $U$  is finite, then we can easily and effectively represent sets by bit vectors
- Specifically, we force an ordering on the objects, say:

$$U = \{a_1, a_2, \dots, a_n\}$$

- For a set  $A \subseteq U$ , a bit vector can be defined as, for  $i=1, 2, \dots, n$ 
  - $b_i=0$  if  $a_i \notin A$
  - $b_i=1$  if  $a_i \in A$

# Computer Representation of Sets (2)

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- Examples
  - Let  $U=\{0,1,2,3,4,5,6,7\}$  and  $A=\{0,1,6,7\}$
  - The bit vector representing A is: 1100 0011
  - How is the empty set represented?
  - How is U represented?
- Set operations become trivial when sets are represented by bit vectors
  - Union is obtained by making the bit-wise OR
  - Intersection is obtained by making the bit-wise AND

# Computer Representation of Sets (3)

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- Let  $U=\{0,1,2,3,4,5,6,7\}$ ,  $A=\{0,1,6,7\}$ ,  $B=\{0,4,5\}$
- What is the bit-vector representation of  $B$ ?
- Compute, bit-wise, the bit-vector representation of  $A \cap B$
- Compute, bit-wise, the bit-vector representation of  $A \cup B$
- What sets do these bit vectors represent?

# Programming Question

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- Using bit vector, we can represent sets of cardinality equal to the size of the vector
- What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?
- What data structure could we use?