Sequences & Summations

Section 2.4 of Rosen

Spring 2012

CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Questions: Piazza

Outline

Although you are (more or less) familiar with sequences and summations, we give a quick review

- Sequences
  - Definition, 2 examples
- Progressions: Special sequences
  - Geometric, arithmetic
- Summations
  - Careful when changing lower/upper limits
- Series: Sum of the elements of a sequence
  - Examples, infinite series, convergence of a geometric series

Sequences

- **Definition**: A <u>sequence</u> is a function from a subset of integers to a set S. We use the notation(s):

$$\{a_n\} \quad \{a_n\}_n^{\infty} \quad \{a_n\}_{n=0}^{\infty}$$

- Each a_n is called the n^{th} <u>term</u> of the sequence
- We rely on the context to distinguish between a sequence and a set, although they are distinct structures

Sequences: Example 1

- Consider the sequence

$$\{(1 + 1/n)^n\}_{n=1}^{\infty}$$

- The terms of the sequence are:

$$a_1 = (1 + 1/1)^1 = 2.00000$$
$$a_2 = (1 + 1/2)^2 = 2.25000$$
$$a_3 = (1 + 1/3)^3 = 2.37037$$
$$a_4 = (1 + 1/4)^4 = 2.44140$$
$$a_5 = (1 + 1/5)^5 = 2.48832$$

- What is this sequence?
- The sequence corresponds to $\lim_{n\to\infty}\{(1 + 1/n)^n\}_{n=1}^{\infty} = e = 2.71828..$

Sequences: Example 2

- The sequence: $\{h_n\}_{n=1}^{\infty} = 1/n$

  is known as the <u>harmonic</u> sequence
- The sequence is simply:

  $$1, 1/2, 1/3, 1/4, 1/5, \ldots$$
- This sequence is particularly interesting because its summation is divergent:

  $$\Sigma_{n=1}^{\infty} (1/n) = \infty$$

Progressions: Geometric

- **Definition**: A <u>geometric progression</u> is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$$

  Where:
  - $a \in R$ is called the <u>initial term</u>
  - $r \in R$ is called the <u>common ratio</u>
- A geometric progression is a <u>discrete</u> analogue of the exponential function

$$f(x) = ar^x$$

Geometric Progressions: Examples

- A common geometric progression in Computer Science is:

$$\{a_n\} = 1/2^n$$

  with a=1 and r=1/2

- Give the initial term and the common ratio of
  - $\{b_n\}$ with $b_n = (-1)^n$
  - $\{c_n\}$ with $c_n = 2(5)^n$
  - $\{d_n\}$ with $d_n = 6(1/3)^n$

Progressions: Arithmetic

- **Definition**: An <u>arithmetric progression</u> is a sequence of the form

  a, a+d, a+2d, a+3d, …, a+nd, …

  Where:
  - $a \in R$ is called the <u>initial term</u>
  - $d \in R$ is called the <u>common difference</u>
- An arithmetic progression is a <u>discrete</u> analogue of the linear function

  $$f(x) = dx+a$$

Arithmetic Progressions: Examples

- Give the initial term and the common difference of
  - $\{s_n\}$ with $s_n = -1 + 4n$
  - $\{t_n\}$ with $s_n = 7 - 3n$

More Examples

- Table 1 on Page 162 (Rosen) has some useful sequences:

$$\{n^2\}_{n=1}^{\infty}, \{n^3\}_{n=1}^{\infty}, \{n^4\}_{n=1}^{\infty}, \{2^n\}_{n=1}^{\infty}, \{3^n\}_{n=1}^{\infty}, \{n!\}_{n=1}^{\infty}$$

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Summations (1)

- You should be by now familiar with the summation notation:

$$\Sigma_{j=m}^{n} (a_j) = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

  Here
  - j is the index of the summation
  - m is the lower limit
  - n is the upper limit
- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

$$\Sigma_{j=1}^{n} (a_j) = \Sigma_{i=o}^{n-1} (a_{i+1})$$

Summations (2)

- Sometimes we can express a summation in <u>closed</u> form, as for geometric series
- **Theorem**: For a, $r \in R$, $r \neq 0$

$$\Sigma_{i=0}^{n} (ar^{i}) = \begin{cases} (ar^{n+1}-a)/(r-1) & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Closed form = analytical expression using a bounded number of well-known functions, does not involved an infinite series or use of recursion

Summations (3)

- Double summations often arise when analyzing an algorithm

$\Sigma_{i=1}^{n} \Sigma_{j=1}^{i}(a_j) = a_1+$

a_1+a_2+

$a_1+a_2+a_3+$

…

$a_1+a_2+a_3+\ldots+a_n$

- Summations can also be indexed over elements in a set:

$$\Sigma_{s \in S} f(s)$$

- Table 2 on Page 166 (Rosen) has very useful summations. Exercises 2.4.30—34 (edition 7th) are great material to practice on.

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Series

- When we take the sum of a sequence, we get a series
- We have already seen a closed form for geometric series
- Some other useful closed forms include the following:
  - $\Sigma_{i=k}^{u} 1 = u-k+1$, for $k \leq u$
  - $\Sigma_{i=0}^{n} i = n(n+1)/2$
  - $\Sigma_{i=0}^{n} (i^2) = n(n+1)(2n+1)/6$
  - $\Sigma_{i=0}^{n} (i^{k}) \approx n^{k+1}/(k+1)$

Infinite Series

- Although we will mostly deal with finite series (i.e., an upper limit of n for fixed integer), inifinite series are also useful
- Consider the following geometric series:
  - $\Sigma_{n=0}^{\infty} (1/2^n) = 1 + 1/2 + 1/4 + 1/8 + \ldots$ converges to 2
  - $\Sigma_{n=0}^{\infty} (2^n) = 1 + 2 + 4 + 8 + \ldots$ does not converge
- However note: $\Sigma_{n=0}^{n} (2^n) = 2^{n+1} - 1$ (a=1,r=2)

Infinite Series: Geometric Series

- In fact, we can generalize that fact as follows
- **Lemma**: A geometric series converges <u>if and only if</u> the absolute value of the common ratio is less than 1

When $|r|<1$,

$$\lim_{n\to\infty}\Sigma_{i=0}^{n} (ar^i) = \lim_{n\to\infty}\Sigma_{i=0}^{n} (ar^{n+1}-a)/(r-1)$$
$$= a/(1-r)$$