

# Partial Orders

## Section 9.6 of Rosen

Spring 2012

CSCE 235 Introduction to Discrete Structures

Course web-page: [cse.unl.edu/~cse235](http://cse.unl.edu/~cse235)

**Questions:** Piazza

# Outline

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- Motivating example
- Definitions
  - Partial ordering, comparability, total ordering, well ordering
- Principle of well-ordered induction
- Lexicographic orderings
- Hasse Diagrams
- Extremal elements
- Lattices
- Topological Sorting

# Motivating Example (1)

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- Consider the renovation of Avery Hall. In this process several tasks were undertaken
  - Remove Asbestos
  - Replace windows
  - Paint walls
  - Refinish floors
  - Assign offices
  - Move in office furniture

# Motivating Example (2)

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- Clearly, some things had to be done before others could begin
  - Asbestos had to be removed before anything (except assigning offices)
  - Painting walls had to be done before refinishing floors to avoid ruining them, etc.
- On the other hand, several things could be done concurrently:
  - Painting could be done while replacing the windows
  - Assigning offices could be done at anytime before moving in office furniture
- This scenario can be nicely modeled using partial orderings

# Partial Orderings: Definitions

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- **Definitions:**

- A relation  $R$  on a set  $S$  is called a partial order if it is
  - Reflexive
  - Antisymmetric
  - Transitive
- A set  $S$  together with a partial ordering  $R$  is called a partially ordered set (poset, for short) and is denoted  $(S,R)$

- Partial orderings are used to give an order to sets that may not have a natural one
- In our renovation example, we could define an ordering such that  $(a,b) \in R$  if 'a must be done before b can be done'

# Partial Orderings: Notation

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- We use the notation:
  - $a \preceq b$ , when  $(a,b) \in R$  `\preccurlyeq`
  - $a \prec b$ , when  $(a,b) \in R$  and  $a \neq b$  `\prec`
- The notation  $\prec$  is not to be mistaken for “less than” ( $\preceq$  versus  $\leq$ )
- The notation  $\prec$  is used to denote any partial ordering

# Comparability: Definition

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- **Definition:**

- The elements  $a$  and  $b$  of a poset  $(S, \preceq)$  are called comparable if either  $a \preceq b$  or  $b \preceq a$ .

- When for  $a, b \in S$ , we have neither  $a \preceq b$  nor  $b \preceq a$ , we say that  $a, b$  are incomparable

- Consider again our renovation example

- Remove Asbestos  $\prec a_i$  for all activities  $a_i$  except assign offices

- Paint walls  $\prec$  Refinish floors

- Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices

# Total orders: Definition

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- **Definition:**

- If  $(S, \preceq)$  is a poset and every two elements of  $S$  are comparable,  $S$  is called a totally ordered set.
- The relation  $\preceq$  is said to be a total order

- **Example**

- The relation “less than or equal to” over the set of integers  $(\mathbb{Z}, \leq)$  since for every  $a, b \in \mathbb{Z}$ , it must be the case that  $a \leq b$  or  $b \leq a$
- What happens if we replace  $\leq$  with  $<$ ?

The relation  $<$  is not reflexive, and  $(\mathbb{Z}, <)$  is not a poset



# Well Orderings: Definition

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- **Definition:**  $(S, \preccurlyeq)$  is a well-ordered set if
  - It is a poset
  - Such that  $\preccurlyeq$  is a total ordering and
  - Such that every non-empty subset of  $S$  has a least element
- Example
  - The natural numbers along with  $\leq$ ,  $(\mathbb{N}, \leq)$ , is a well-ordered set since any nonempty subset of  $\mathbb{N}$  has a least element and  $\leq$  is a total ordering on  $\mathbb{N}$
  - However,  $(\mathbb{Z}, \leq)$  is not a well-ordered set
    - Why?  $\mathbb{Z}^- \subset \mathbb{Z}$  but does not have a least element
    - Is it totally ordered? Yes

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- Hasse Diagrams
- Extremal elements
- Lattices
- Topological Sorting

# Principle of Well-Ordered Induction

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- Well-ordered sets are the basis of the proof technique known as induction (more when we cover Chapter 3)
- **Theorem: Principle of Well-Ordered Induction**

Given  $S$  is a well-ordered set.  $P(x)$  is true for all  $x \in S$  if

(**Basis Step:**  $P(x_0)$  is true for the least element in  $S$  and)

**Inductive Step:** For every  $y \in S$  if  $P(x)$  is true for all  $x \prec y$ , then  $P(y)$  is true

# Principle of Well-Ordered Induction: Proof

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**Proof:**  $(S \text{ well ordered}) \wedge (\text{Basis Step}) \wedge (\text{Induction Step}) \Rightarrow \forall x \in S, P(x)$

- Suppose that it is not the case the  $P(x)$  holds for all  $x \in S$ 
  - $\Rightarrow \exists y P(y)$  is false
  - $\Rightarrow A = \{ x \in S \mid P(x) \text{ is false} \}$  is not empty
- $S$  is well ordered  $\Rightarrow A$  has a least element  $a$
- Since  $P(x_0)$  is true and  $P(a)$  is false  $\Rightarrow a \neq x_0$
- $P(x)$  holds for all  $x \in S$  and  $x \prec a$ , then  $P(a)$  holds by the induction step
- This yields a contradiction **QED**

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- **Lexicographic orderings**
  - **Idea, on  $A_1 \times A_2$ ,  $A_1 \times A_2 \times \dots \times A_n$ ,  $S^t$  (strings)**
- Hasse Diagrams
- Extremal elements
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# Lexicographic Orderings: Idea

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- Lexicographic ordering is the same as any dictionary or phone-book ordering:
  - We use alphabetic ordering
    - Starting with the first character in the string
    - Then the next character, if the first was equal, etc.
  - If a word is shorter than the other, than we consider that the ‘no character’ of the shorter word to be less than ‘a’

# Lexicographic Orderings on $A_1 \times A_2$

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- Formally, lexicographic ordering is defined by two other orderings
- **Definition:** Let  $(A_1, \preceq_1)$  and  $(A_2, \preceq_2)$  be two posets. The lexicographic ordering  $\prec$  on the Cartesian product  $A_1 \times A_2$  is defined by
$$(a_1, a_2) \prec (a'_1, a'_2) \text{ if } (a_1 \prec_1 a'_1) \text{ or } (a_1 = a'_1 \text{ and } a_2 \prec_2 a'_2)$$
- If we add equality to the lexicographic ordering  $\prec$  on  $A_1 \times A_2$ , we obtain a partial ordering

# Lexicographic Ordering on $A_1 \times A_2 \times \dots \times A_n$

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- Lexicographic ordering generalizes to the Cartesian Product of  $n$  set in a natural way
- Define  $\prec$  on  $A_1 \times A_2 \times \dots \times A_n$  by

$$(a_1, a_2, \dots, a_n) \prec (b_1, b_2, \dots, b_n)$$

If  $a_1 \prec b_1$  or if there is an integer  $i > 0$  such that

$$a_1 = b_1, a_2 = b_2, \dots, a_i = b_i \text{ and } a_{i+1} \prec b_{i+1}$$



# Lexicographic Ordering on Strings

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- Consider the two non-equal strings  $a_1a_2\dots a_m$  and  $b_1b_2\dots b_n$  on a poset  $(S^t, \preceq)$
- Let
  - $t = \min(n, m)$
  - $\prec$  be the lexicographic ordering on  $S^t$
- $a_1a_2\dots a_m$  is less than  $b_1b_2\dots b_n$  if and only if
  - $(a_1, a_2, \dots, a_t) \prec (b_1, b_2, \dots, b_t)$  or
  - $(a_1, a_2, \dots, a_t) = (b_1, b_2, \dots, b_t)$  and  $m < n$

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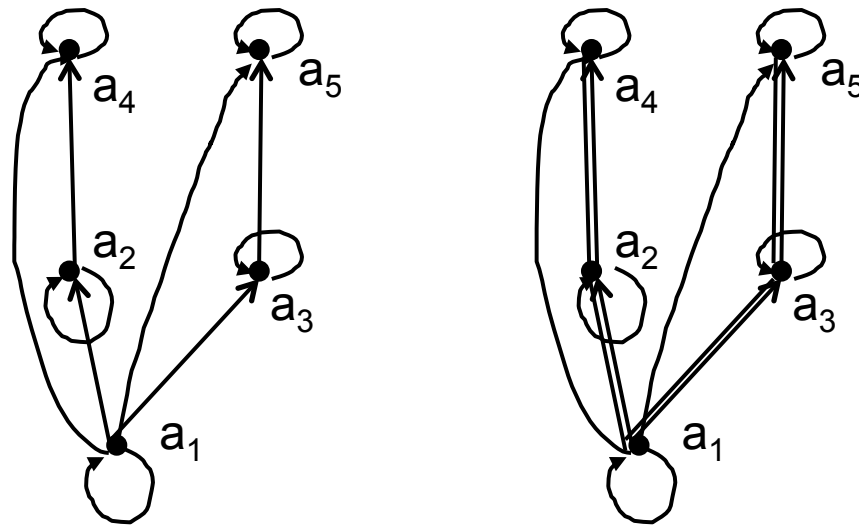
# Hasse Diagrams

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- Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  - Consider the digraph representation of a partial order
  - Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  - Thus, we can simplify the graph as follows
    - Remove all self loops
    - Remove all transitive edges
    - Remove directions on edges assuming that they are oriented upwards
  - The resulting diagram is far simpler

# Hasse Diagram: Example

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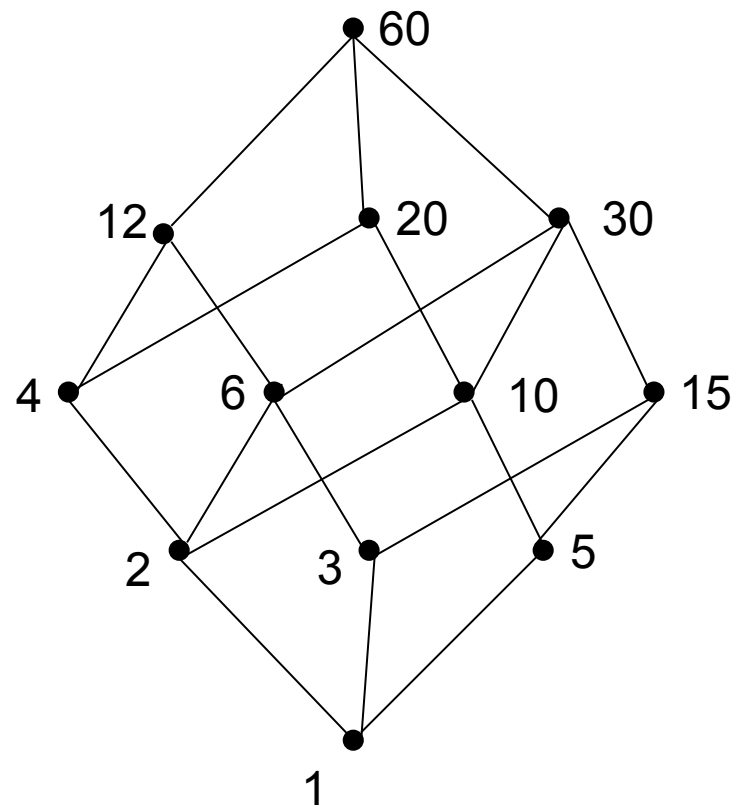
# Hasse Diagrams: Example (1)

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- Of course, you need not always start with the complete relation in the partial order and then trim everything.
- Rather, you can build a Hasse Diagram directly from the partial order
- Example: Draw the Hasse Diagram
  - for the following partial ordering:  $\{(a,b) \mid a|b\}$
  - on the set  $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
  - (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)

# Hasse Diagram: Example (2)

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# Extremal Elements: Summary

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We will define the following terms:

- A maximal/minimal element in a poset  $(S, \preceq)$
- The maximum (greatest)/minimum (least) element of a poset  $(S, \preceq)$
- An upper/lower bound element of a subset  $A$  of a poset  $(S, \preceq)$
- The greatest lower/least upper bound element of a subset  $A$  of a poset  $(S, \preceq)$



# Extremal Elements: Maximal

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- **Definition:** An element  $a$  in a poset  $(S, \preceq)$  is called maximal if it is not less than any other element in  $S$ . That is:  $\neg(\exists b \in S (a \prec b))$
- If there is one unique maximal element  $a$ , we call it the maximum element (or the greatest element)

# Extremal Elements: Minimal

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- **Definition:** An element  $a$  in a poset  $(S, \preceq)$  is called minimal if it is not greater than any other element in  $S$ . That is:  $\neg(\exists b \in S (b \prec a))$
- If there is one unique minimal element  $a$ , we call it the minimum element (or the least element)

# Extremal Elements: Upper Bound

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- **Definition:** Let  $(S, \preceq)$  be a poset and let  $A \subseteq S$ . If  $u$  is an element of  $S$  such that  $a \preceq u$  for all  $a \in A$  then  $u$  is an upper bound of  $A$
- An element  $x$  that is an upper bound on a subset  $A$  and is less than all other upper bounds on  $A$  is called the least upper bound on  $A$ . We abbreviate it as lub.

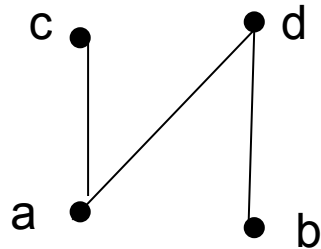
# Extremal Elements: Lower Bound

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- **Definition:** Let  $(S, \preceq)$  be a poset and let  $A \subseteq S$ . If  $l$  is an element of  $S$  such that  $l \preceq a$  for all  $a \in A$  then  $l$  is an lower bound of  $A$
- An element  $x$  that is a lower bound on a subset  $A$  and is greater than all other lower bounds on  $A$  is called the greatest lower bound on  $A$ . We abbreviate it glb.

# Extremal Elements: Example 1

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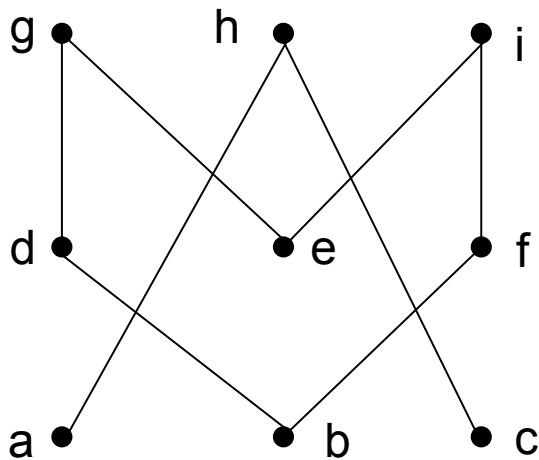
What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

# Extremal Elements: Example 2

Give lower/upper bounds  
& glb/lub of the sets:

$\{d,e,f\}$ ,  $\{a,c\}$  and  $\{b,d\}$



$\{d,e,f\}$

- Lower bounds:  $\emptyset$ , thus no glb
- Upper bounds:  $\emptyset$ , thus no lub

$\{a,c\}$

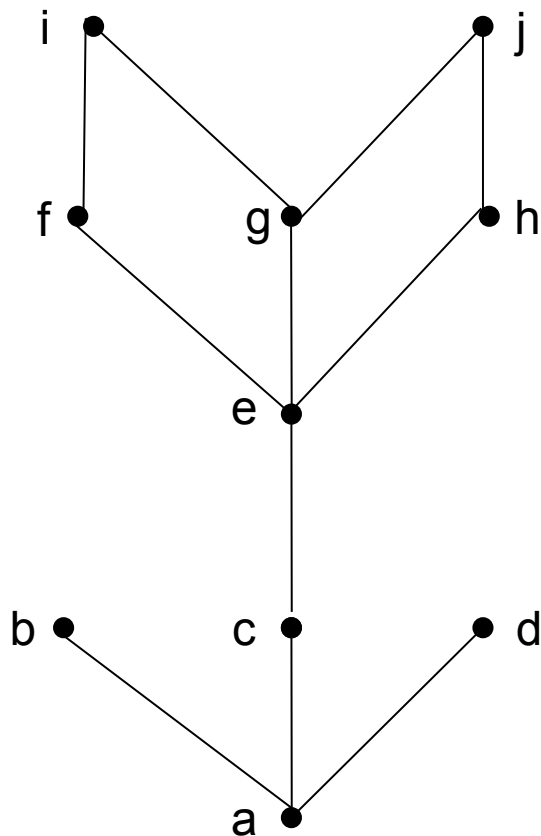
- Lower bounds:  $\emptyset$ , thus no glb
- Upper bounds:  $\{h\}$ , lub: h

$\{b,d\}$

- Lower bounds:  $\{b\}$ , glb: b
- Upper bounds:  $\{d,g\}$ , lub: d because  $d \prec g$

# Extremal Elements: Example 3

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- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
  - Lower bounds: {a,c}, thus glb is c
  - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - Lower bounds: {a}, thus glb is a
  - Upper bounds:  $\emptyset$ , thus lub DNE

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# Lattices

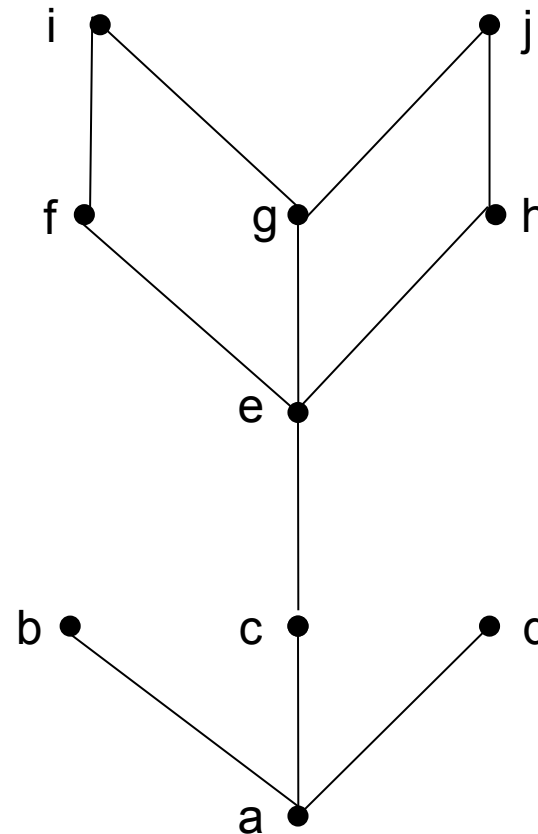
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- A special structure arises when every pair of elements in a poset has an lub and a glb
- **Definition:** A lattice is a partially ordered set in which every pair of elements has both
  - a least upper bound and
  - a greatest lower bound

# Lattices: Example 1

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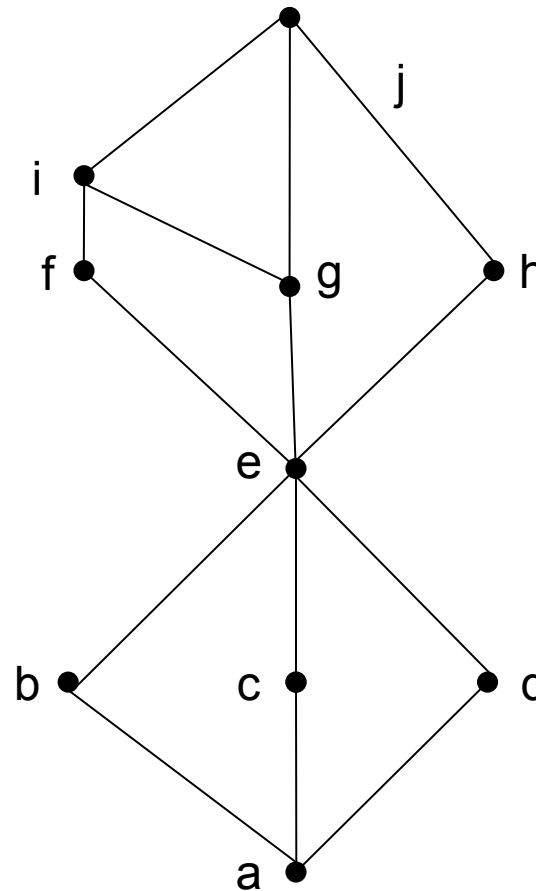
- Is the example from before a lattice?
- **No, because the pair  $\{b,c\}$  does not have a least upper bound**



# Lattices: Example 2

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- What if we modified it as shown here?
- **Yes, because for any pair, there is an lub & a glb**



# Lattices: Example 3

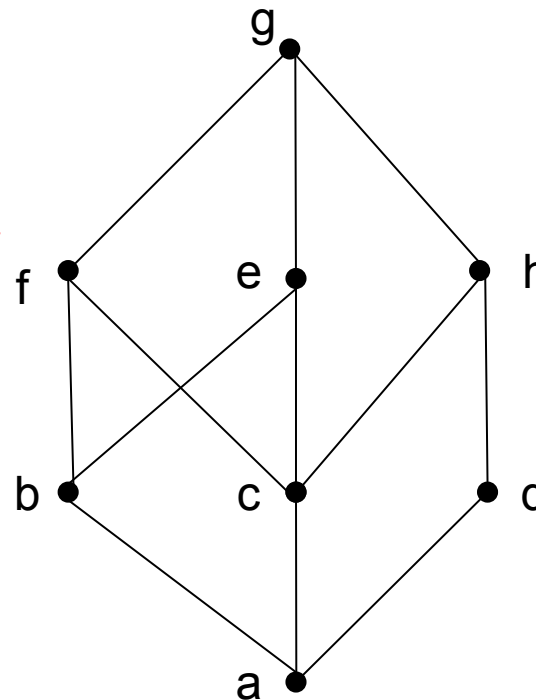
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- Is this example a lattice?

**No!**

- The lower bound of  $A=\{e,f\}$  is  $\{a,b,c\}$
- However,  $A$  has no glb

**Similarly,  $B=\{b,c\}$  has no ulb**



# A Lattice Or Not a Lattice?

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- To show that a partial order is not a lattice, it suffices to find a pair that does not have a lub or a glb (i.e., a counter-example)
- For a pair not to have a lub/glb, the elements of the pair must first be incomparable (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice

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# Topological Sorting

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- Let us return to the introductory example of Avery Hall renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule
- That is, given a partial order, we would like to transform it into a total order that is compatible with the partial order
- A total order is compatible if it does not violate any of the original relations in the partial order
- Essentially, we are simply imposing an order on incomparable elements in the partial order

# Topological Sorting: Preliminaries (1)

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- Before we give the algorithm, we need some tools to justify its correctness
- **Fact:** Every finite, nonempty poset  $(S, \preceq)$  has a minimal element
- We will prove the above fact by a form of *reductio ad absurdum*



# Topological Sorting: Preliminaries (2)

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- **Proof:**

- Assume, to the contrary, that a nonempty finite poset  $(S, \preceq)$  has no minimal element. In particular, assume that  $a_1$  is not a minimal element.
- Assume, w/o loss of generality, that  $|S|=n$
- If  $a_1$  is not minimal, then there exists  $a_2$  such that  $a_2 \prec a_1$
- But  $a_2$  is also not minimal because of the above assumption
- Therefore, there exists  $a_3$  such that  $a_3 \prec a_2$ . This process proceeds until we have the last element  $a_n$ . Thus,  $a_n \prec a_{n-1} \prec \dots \prec a_2 \prec a_1$
- Finally, by definition  $a_n$  is the minimal element **QED**

# Topological Sorting: Intuition

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- The idea of topological sorting is
  - We start with a poset  $(S, \preceq)$
  - We remove a minimal element, choosing arbitrarily if there is more than one. Such an element is guaranteed to exist by the previous fact
  - As we remove each minimal element, one at a time, the set  $S$  shrinks
  - Thus we are guaranteed that the algorithm will terminate in a finite number of steps
  - Furthermore, the order in which the elements are removed is a total order:  $a_1 \prec a_2 \prec \dots \prec a_{n-1} \prec a_n$
- Now, we can give the algorithm itself

# Topological Sorting: Algorithm

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*Input:*  $(S, \preceq)$  a poset with  $|S|=n$

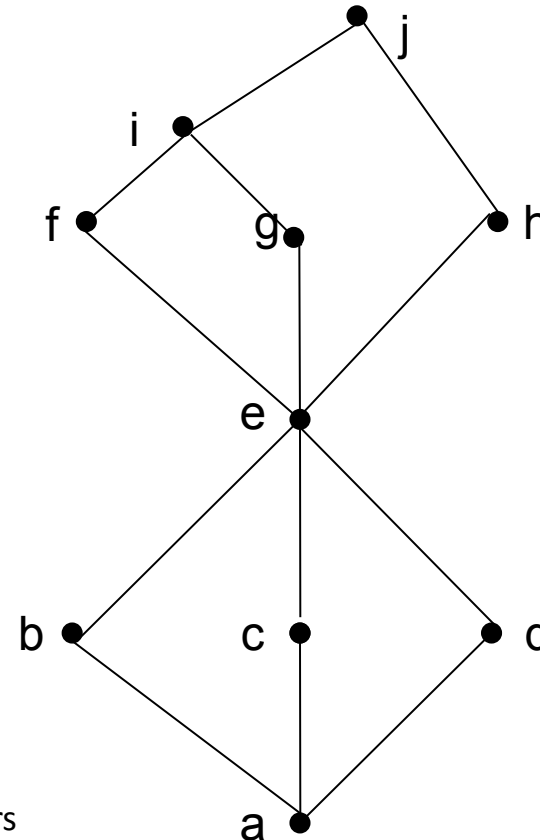
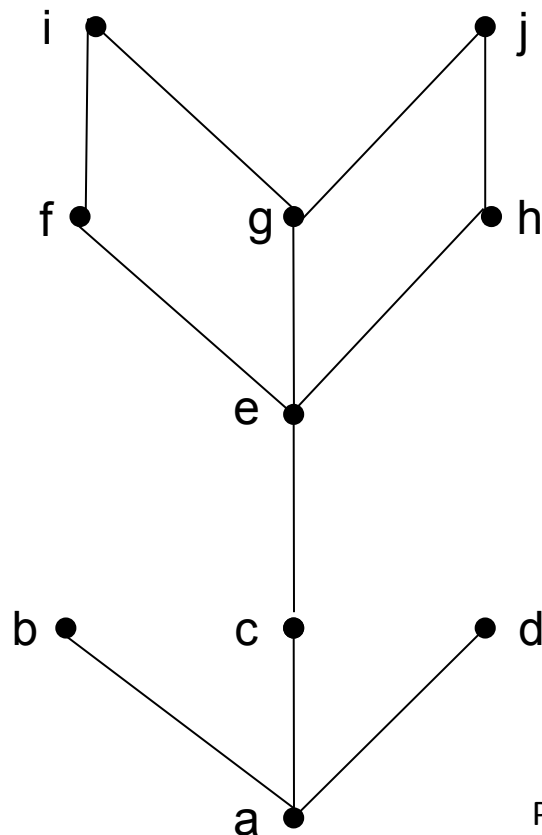
*Output:* A total ordering  $(a_1, a_2, \dots, a_n)$

1.  $k \leftarrow 1$
2. **While**  $S$  **Do**
3.    $a_k \leftarrow$  a minimal element in  $S$
4.    $S \leftarrow S \setminus \{a_k\}$
5.    $k \leftarrow k+1$
6. **End**
7. **Return**  $(a_1, a_2, \dots, a_n)$

# Topological Sorting: Example

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- Find a compatible ordering (topological ordering) of the poset represented by the Hasse diagrams below



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  - Idea, on  $A_1 \times A_2$ ,  $A_1 \times A_2 \times \dots \times A_n$ ,  $S^t$  (strings)
- Hasse Diagrams
- Extremal elements
  - Minimal/minimum, maximal/maximum, glb, lub
- Lattices
- Topological Sorting