Master Theorem

Section 8.3 of Rosen
Spring 2012
CSCE 235 Introduction to Discrete Structures
Course web-page: cse.unl.edu/~cse235
Questions: Piazza
Outline

• Motivation
• The Master Theorem
  – Pitfalls
  – 3 examples
• 4\textsuperscript{th} Condition
  – 1 example
Motivation: Asymptotic Behavior of Recursive Algorithms

• When analyzing algorithms, recall that we only care about the asymptotic behavior
• Recursive algorithms are no different
• Rather than solving exactly the recurrence relation associated with the cost of an algorithm, it is sufficient to give an asymptotic characterization
• The main tool for doing this is the master theorem
Outline

• Motivation

• The Master Theorem
  – Pitfalls
  – 3 examples

• 4\textsuperscript{th} Condition
  – 1 example
Master Theorem

• Let $T(n)$ be a monotonically increasing function that satisfies

$$T(n) = a \cdot T(n/b) + f(n)$$

$T(1) = c$

where $a \geq 1$, $b \geq 2$, $c > 0$. If $f(n)$ is $\Theta(n^d)$ where $d \geq 0$ then

$$T(n) = \begin{cases} 
\Theta(n^d) & \text{if } a < b^d \\
\Theta(n^d \log n) & \text{if } a = b^d \\
\Theta(n^{\log_b a}) & \text{if } a > b^d 
\end{cases}$$
Master Theorem: Pitfalls

• You cannot use the Master Theorem if
  – $T(n)$ is not monotone, e.g. $T(n) = \sin(x)$
  – $f(n)$ is not a polynomial, e.g., $T(n) = 2T(n/2) + 2^n$
  – $b$ cannot be expressed as a constant, e.g.
    $$T(n) = T(\sqrt{n})$$

• Note that the Master Theorem does not solve the recurrence equation

• Does the base case remain a concern?
Master Theorem: Example 1

• Let $T(n) = T(n/2) + \frac{1}{2} n^2 + n$. What are the parameters?
  
  $a = 1$
  
  $b = 2$
  
  $d = 2$

  Therefore, which condition applies?

  $1 < 2^2$, case 1 applies

• We conclude that

  \[ T(n) \in \Theta(n^d) = \Theta(n^2) \]
Master Theorem: Example 2

• Let $T(n) = 2T(n/4) + \sqrt{n} + 42$. What are the parameters?
  
  $a = 2$
  
  $b = 4$
  
  $d = 1/2$
  
  Therefore, which condition applies?
  
  $2 = 4^{1/2}$, case 2 applies
  
• We conclude that
  
  $T(n) \in \Theta(n^d \log n) = \Theta(\log n \sqrt{n})$
Master Theorem: Example 3

• Let $T(n) = 3T(n/2) + 3/4n + 1$. What are the parameters?
  
  \[ a = 3 \]
  \[ b = 2 \]
  \[ d = 1 \]

  Therefore, which condition applies?
  
  $3 > 2^1$, case 3 applies

• We conclude that

  \[ T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3}) \]

• Note that $\log_2 3 \approx 1.584...$, can we say that $T(n) \in \Theta(n^{1.584})$?

  No, because $\log_2 3 \approx 1.5849...$ and $n^{1.584} \not\in \Theta(n^{1.5849})$
Outline

• Motivation
• The Master Theorem
  – Pitfalls
  – 3 examples
• 4th Condition
  – 1 example
‘Fourth’ Condition

• Recall that we cannot use the Master Theorem if \( f(n) \), the non-recursive cost, is not a polynomial

• There is a limited 4\(^{th}\) condition of the Master Theorem that allows us to consider polylogarithmic functions

• **Corollary:** If \( f(n) \in \Theta(n^{\log_b a} \log^k n) \) for some \( k \geq 0 \) then

\[
T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)
\]

• This final condition is fairly limited and we present it merely for sake of completeness.. Relax 😊
‘Fourth’ Condition: Example

- Say we have the following recurrence relation
  \[ T(n) = 2 \cdot T(n/2) + n \cdot \log n \]

- Clearly, \( a = 2 \), \( b = 2 \), but \( f(n) \) is not a polynomial. However, we have \( f(n) \in \Theta(n \log n) \), \( k = 1 \)

- Therefore by the 4\(^{th}\) condition of the Master Theorem we can say that

\[
T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n^{\log_2 2 \log^2 n}) = \Theta(n \log^2 n)
\]
Summary

• Motivation
• The Master Theorem
  – Pitfalls
  – 3 examples
• 4th Condition
  – 1 example