Predicate Logic and Quantifies

Sections 1.4, and 1.5 of Rosen
Spring 2012
CSCE 235 Introduction to Discrete Structures
Course web-page: cse.unl.edu/~cse235
All questions: Piazza
LaTeX

• Using the package: `\usepackage{amssymb}`
  – Set of natural numbers: `$\mathbb{N}$`
  – Set of integer numbers: `$\mathbb{Z}$`
  – Set of rational numbers: `$\mathbb{Q}$`
  – Set of real numbers: `$\mathbb{R}$`
  – Set of complex numbers: `$\mathbb{C}$`
Outline

• Introduction
• Terminology:
  – Propositional functions; arguments; arity; universe of discourse
• Quantifiers
  – Definition; using, mixing, negating them
• Logic Programming (Prolog)
• Transcribing English to Logic
• More exercises
Introduction

• Consider the statements:
  \[ x > 3, \ x = y + 3, \ x + y = z \]
• The symbols \( >, +, = \) denote relations between \( x \) and 3, \( x, y, \) and 4, and \( x, y, \) and \( z, \) respectively
• These relations may hold or not hold depending on the values that \( x, y, \) and \( z \) may take.
• A **predicate** is a property that is affirmed or denied about the subject (in logic, we say ‘**variable**’ or ‘**argument**’) of a statement
• Consider the statement: ‘\( x \) is greater than 3’
  – ‘\( x \)’ is the subject
  – ‘is greater than 3’ is the predicate
Propositional Functions (1)

• To write in Predicate Logic ‘x is greater than 3’
  – We introduce a functional symbol for the predicate and
  – Put the subject as an argument (to the functional symbol): $P(x)$

• Terminology
  – $P(x)$ is a statement
  – $P$ is a predicate or propositional function
  – $x$ as an argument
  – $P$(Bob) is a proposition
Propositional Functions (2)

• Examples:
  – Father(x): unary predicate
  – Brother(x,y): binary predicate
  – Sum(x,y,z): ternary predicate
  – P(x,y,z,t): n-ary predicate
Propositional Functions (3)

• **Definition:** A statement of the form $P(x_1, x_2, ..., x_n)$ is the value of the propositional symbol $P$.

• Here: $(x_1, x_2, ..., x_n)$ is an $n$-tuple and $P$ is a predicate

• We can think of a propositional function as a function that
  – Evaluates to true or false
  – Takes one or more arguments
  – Expresses a predicate involving the argument(s)
  – Becomes a proposition when values are assigned to the arguments
Propositional Functions: Example

• Let $Q(x, y, z)$ denote the statement ‘$x^2 + y^2 = z^2$’
  
  – What is the truth value of $Q(3, 4, 5)$?
    
    $Q(3, 4, 5)$ is true
  
  – What is the truth value of $Q(2, 2, 3)$?
    
    $Q(2, 3, 3)$ is false
  
  – How many values of $(x, y, z)$ make the predicate true?
    
    There are infinitely many values that make the proposition true, how many right triangles are there?
Universe of Discourse

• Consider the statement ‘x>3’, does it make sense to assign to x the value ‘blue’?
• Intuitively, the **universe of discourse** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
• What would be the universe of discourse for the propositional function below be:
  
  $\text{EnrolledCSE235}(x) = 'x \text{ is enrolled in CSE235'}$
Universe of Discourse: Multivariate functions

- Each variable in an $n$-tuple (i.e., each argument) may have a different universe of discourse

- Consider an $n$-ary predicate $P$:

  $P(r,g,b,c) = \text{‘The rgb-values of the color } c \text{ is } (r,g,b)\text{’}$

- Example, what is the truth value of
  - $P(255,0,0,\text{red})$
  - $P(0,0,255,\text{green})$

- What are the universes of discourse of $(r,g,b,c)$?
Alert

• Propositional Logic (PL)
  – Sentential logic
  – Boolean logic
  – Zero order logic

• First Order Logic (FOL)
  – Predicate logic (PL)
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Quantifiers: Introduction

• The statement ‘x>3’ is not a proposition
• It becomes a proposition
  – When we assign values to the argument: ‘4>3’ is true, ‘2<3’ is false, or
  – When we quantify the statement
• Two quantifiers
  – Universal quantifier $\forall$
    the proposition is true for all possible values in the universe of discourse
  – Existential quantifier $\exists$
    the proposition is true for some value(s) in the universe of discourse
Universal Quantifier: Definition

**Definition:** The universal quantification of a predicate $P(x)$ is the proposition ‘$P(x)$ is true for all values of $x$ in the universe of discourse.’ We use the notation: $\forall x P(x)$, which is read ‘for all $x$’.

- If the universe of discourse is finite, say $\{n_1,n_2,\ldots,n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements

  $$\forall x P(x) \iff P(n_1) \land P(n_2) \land \ldots \land P(n_k)$$
Universal Quantifier: Example 1

• Let
  – $P(x)$: ‘$x$ must take a discrete mathematics course’ and
  – $Q(x)$: ‘$x$ is a CS student.’

• The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.

• Express the statements:
  – “Every CS student must take a discrete mathematics course.”
    \[
    \forall x \quad Q(x) \rightarrow P(x)
    \]
  – “Everybody must take a discrete mathematics course or be a CS student.”
    \[
    \forall x \quad ( P(x) \lor Q(x) )
    \]
  – “Everybody must take a discrete mathematics course and be a CS student.”
    \[
    \forall x \quad ( P(x) \land Q(x) )
    \]

Are these statements true or false at UNL?
Universal Quantifier: Example 2

• Express in FOL the statement
  ‘for every x and every y, x+y>10’

• Answer:
  1. Let $P(x,y)$ be the statement $x+y>10$
  2. Where the universe of discourse for $x, y$ is the set of integers
  3. The statement is: $\forall x \forall y \, P(x,y)$

• Shorthand: $\forall x, y \, P(x,y)$
Existential Quantifier: Definition

- **Definition**: The existential quantification of a predicate $P(x)$ is the proposition ‘There exists a value $x$ in the universe of discourse such that $P(x)$ is true’
  
  - Notation: $\exists x \ P(x)$
  
  - Reads: ‘there exists $x$’

- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of the propositions over all the elements

  $$\exists x \ P(x) \iff P(n_1) \lor P(n_2) \lor \ldots \lor P(n_k)$$
Existential Quantifier: Example 1

• Let $P(x,y)$ denote the statement ‘$x+y=5$’
• What does the expression $\exists x \exists y P(x,y)$ mean?
• Which universe(s) of discourse make it true?
Existential Quantifier: Example 2

• Express formally the statement
  ‘there exists a real solution to $ax^2+bx-c=0$’
• Answer:
  1. Let $P(x)$ be the statement $x = (-b\pm\sqrt{b^2-4ac})/2a$
  2. Where the universe of discourse for $x$ is the set of real numbers. Note here that $a$, $b$, $c$ are fixed constants.
  3. The statement can be expressed as $\exists x \ P(x)$
• What is the truth value of $\exists x \ P(x)$, where UoD is $R$?
  – It is false. When $b^2<4ac$, there are no real number $x$ that can satisfy the predicate
• What can we do so that $\exists x \ P(x)$ is true?
  – Change the universe of discourse to the complex numbers, $C$
Quantifiers: Truth values

- In general, when are quantified statements true or false?

<table>
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<td>$\forall x \ P(x)$</td>
<td>$P(x)$ is true for every $x$</td>
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Mixing quantifiers (1)

• Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

\[ \forall x \exists y \ P(x,y) \]

is perfectly valid

• Alert:
  – The quantifiers must be read from left to right
  – The order of the quantifiers is important
  – \( \forall x \exists y \ P(x,y) \) is not equivalent to \( \exists y \forall x \ P(x,y) \)
Mixing quantifiers (2)

• Consider
  – $\forall x \ \exists y \ Loves (x,y)$: Everybody loves somebody
  – $\exists y \ \forall x \ Loves(x,y)$: There is someone loved by everyone
• The two expressions do not mean the same thing
• $(\exists y \ \forall x \ Loves(x,y)) \rightarrow (\forall x \ \exists y \ Loves (x,y))$
  but the converse does not hold
• However, you can commute similar quantifiers
  – $\forall x \ \forall y \ P(x,y)$ is equivalent to $\forall y \ \forall x \ P(x,y)$ (thus, $\forall x,y \ P(x,y)$)
  – $\exists x \ \exists y \ P(x,y)$ is equivalent to $\exists y \ \exists x \ P(x,y)$ (thus $\exists x,y \ P(x,y)$)
## Mixing Quantifiers: Truth values

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<td>$\forall x \forall y \ P(x,y)$</td>
<td>$P(x,y)$ is true for every pair $x,y$</td>
<td>There is at least one <em>pair</em> $x,y$ for which $P(x,y)$ is false</td>
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<td>$\forall x \exists y \ P(x,y)$</td>
<td>For every $x$, there is a $y$ for which $P(x,y)$ is true</td>
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Mixing Quantifiers: Example (1)

- Express, in predicate logic, the statement that there is an infinite number of integers
- Answer:

  1. Let $P(x, y)$ be the statement that $x < y$
  2. Let the universe of discourse be the integers, $\mathbb{Z}$
  3. The statement can be expressed by the following

    $\forall x \exists y \ P(x, y)$
Mixing Quantifiers: Example (2)

• Express the *commutative law of addition* for $R$

• We want to express that for every pair of reals, $x, y$, the following holds: $x+y=y+x$

• Answer:
  1. Let $P(x,y)$ be the statement that $x+y$
  2. Let the universe of discourse be the reals, $R$
  3. The statement can be expressed by the following

$$\forall x \forall y \left( P(x,y) \iff P(y,x) \right)$$

Alternatively, $\forall x \forall y \left( x+y = y+x \right)$
Mixing Quantifiers: Example (3)

- Express the multiplicative law for nonzero reals $R \setminus \{0\}$ (i.e., every nonzero real has an inverse)
- We want to express that for every real number $x$, there exists a real number $y$ such that $xy=1$
- Answer:

$$\forall x \exists y \ (xy = 1)$$
Mixing Quantifiers: Example (4)

false mathematical statement

• Does commutativity for substraction hold over the reals?
• That is: does $x-y = y-x$ for all pairs $x,y$ in $R$?
• Express using quantifiers

$$\forall x \forall y \ (x-y = y-x)$$
Mixing Quantifiers: Example (5)

• Express the statement as a logical expression:
  – “There is a number x such that
  – when it is added to any number, the result is that number and
  – if it is multiplied by any number, the result is x”

• Answer:
  • Let $P(x,y)$ be the expression “$x+y=y$”
  • Let $Q(x,y)$ be the expression “$xy=x$”
  • The universe of discourse is $N, Z, R, Q$ (but not $Z^+$)
  • Then the expression is:
    $$\exists x \forall y \ P(x,y) \land Q(x,y)$$
    Alternatively:
    $$\exists x \forall y \ (x+y=y) \land (xy = x)$$
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Binding Variables

• When a quantifier is used on a variable \( x \), we say that \( x \) is **bound**

• If no quantifier is used on a variable in a predicate statement, the variable is called **free**

• Examples
  – In \( \exists x \forall y P(x,y) \), both \( x \) and \( y \) are bound
  – In \( \forall x P(x,y) \), \( x \) is bound but \( y \) is free

• A statement is called a **well-formed formula**, when all variables are properly quantified
Binding Variables: Scope

• The set of all variables bound by a common quantifier is called the scope of the quantifier.

• For example, in the expression $\exists x, y \forall z P(x, y, z, c)$
  – What is the scope of existential quantifier?
  – What is the scope of universal quantifier?
  – What are the bound variables?
  – What are the free variables?
  – Is the expression a well-formed formula?
Negation

• We can use negation with quantified expressions as we used them with propositions

• Lemma: Let $P(x)$ be a predicate. Then the followings hold:

\[\neg(\forall x \ P(x)) \equiv \exists x \ \neg P(x)\]

\[\neg(\exists x \ P(x)) \equiv \forall x \ \neg P(x)\]

• This is essentially the quantified version of De Morgan’s Law (when the universe of discourse is finite, this is exactly De Morgan’s Law)
Negation: Truth

Truth Values of Negated Quantifiers

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<td>( P(x) ) is false for every ( x )</td>
<td>There is an ( x ) for which ( P(x) ) is true</td>
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<td>( \neg \forall x \ P(x) \equiv \exists x \ \neg P(x) )</td>
<td>There is an ( x ) for which ( P(x) ) is false</td>
<td>( P(x) ) is true for every ( x )</td>
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Negation: Example

• Rewrite the following expression, pushing negation inward:
  \[ \neg \forall x (\exists y \forall z P(x,y,z) \land \exists z \forall y P(x,y,z)) \]

• Answer:
  \[ \exists x (\forall y \exists z \neg P(x,y,z) \lor \forall z \exists y \neg P(x,y,z)) \]
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Prolog (1)

• Prolog (Programming in Logic)
  – is a programming language
  – based on (a restricted form of) Predicate Logic
    (a.k.a. Predicate Calculus and FOL)

• It was developed
  – by the logicians of the Artificial Intelligence community
  – for symbolic reasoning
Prolog (2)

- Prolog allows the users to express facts and rules
- Facts are propositional functions:
  - student(mia),
  - enrolled(mia,cse235),
  - instructor(patel,cse235), etc.
- Rules are implications with conjunctions:
  teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:
  ?enrolled(mia,cse235)
  ?enrolled(X,cse476)
  ?teaches(X,mia)
  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5
English into Logic

- Logic is more precise than English
- Transcribing English into Logic and vice versa can be tricky
- When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:
  
  **Use $\forall$ with $\Rightarrow$**
  
  $\forall x \狮子(x) \Rightarrow \text{凶猛}(x)$: Every lion is fierce
  
  $\forall x \狮子(x) \land \text{凶猛}(x)$: Everyone is a lion and everyone is fierce
  
  **Use $\exists$ with $\land$**
  
  $\exists x \狮子(x) \land \text{纯素}(x)$: Holds when you have at least one vegan lion
  
  $\exists x \狮子(x) \Rightarrow \text{纯素}(x)$: Holds when you have vegan people in the universe of discourse (even though there is no vegan lion in the universe of discourse)
More Exercises (1)

• Let $P(x,y)$ denote ‘$x$ is a factor of $y$’ where
  – $x \in \{1,2,3,...\}$ and $y \in \{2,3,4,...\}$
• Let $Q(x,y)$ denote:
  – $\forall x,y \ [P(x,y) \rightarrow (x=y) \lor (x=1)]$
• Question: When is $Q(x,y)$ true?
Alert...

- Some students wonder if:
  \[ \forall x, y \, P(x, y) \equiv (\forall x \, P(x, y)) \land (\forall y \, P(x, y)) \]

- This is certainly not true.
  - In the left-hand side, both \(x, y\) are bound.
  - In the right-hand side,
    - In the first predicate, \(x\) is bound and \(y\) is free
    - In the second predicate, \(y\) is bound and \(x\) is free
    - Thus, the left-hand side is a proposition, but the right-hand side is not. They cannot be equivalent

- All variables that occur in a propositional function must be bound to turn it into a proposition