Predicate Logic and Quantifies

Sections 1.4, and 1.5 of Rosen

Spring 2012

CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

All questions: Piazza

LaTeX

- Using the package: `\usepackage{amssymb}`
  - Set of natural numbers: `$\mathbb{N}$`
  - Set of integer numbers: `$\mathbb{Z}$`
  - Set of rational numbers: `$\mathbb{Q}$`
  - Set of real numbers: `$\mathbb{R}$`
  - Set of complex numbers: `$\mathbb{C}$`

Outline

- Introduction
- Terminology:
  - Propositional functions; arguments; arity; universe of discourse
- Quantifiers
  - Definition; using, mixing, negating them
- Logic Programming (Prolog)
- Transcribing English to Logic
- More exercises

Introduction

- Consider the statements:

$$x>3,\ x=y+3,\ x+y=z$$

- The symbols >, +, = denote relations between *x* and 3, *x*, *y*, and 4, and *x*,*y*, and *z*, respectively
- These relations may hold or not hold depending on the values that *x*, *y*, and *z* may take.
- A <u>**predicate**</u> is a property that is affirmed or denied about the subject (in logic, we say '**variable**' or '**argument**') of a statement
- Consider the statement : '*x* is greater than 3'
  - '*x*' is the subject
  - 'is greater than 3' is the predicate

Propositional Functions (1)

- To write in Predicate Logic '*x* is greater than 3'
  - We introduce a functional symbol for the **predicate** and
  - Put the subject as an **argument** (to the functional symbol): $P(x)$
- Terminology
  - $P(x)$ is a statement
  - P is a predicate or propositional function
  - x as an argument
  - P(Bob) is a proposition

Propositional Functions (2)

- Examples:
  - Father(x): unary predicate
  - Brother(x,y): binary predicate
  - Sum(x,y,z): ternary predicate
  - P(x,y,z,t): n-ary predicate

Propositional Functions (3)

- **Definition:** A statement of the form $P(x_1,x_2,\ldots, x_n)$ is the value of the propositional symbol *P*.
- Here: $(x_1,x_2,\ldots, x_n)$ is an *n*-tuple and *P* is a predicate
- We can think of a propositional function as a function that
  - Evaluates to true or false
  - Takes one or more arguments
  - Expresses a predicate involving the argument(s)
  - Becomes a proposition when values are assigned to the arguments

Propositional Functions: Example

- Let $Q(x,y,z)$ denote the statement '$x^2+y^2=z^2$'
  - What is the truth value of $Q(3,4,5)$?

    $Q(3,4,5)$ is true

  - What is the truth value of $Q(2,2,3)$?

    $Q(2,3,3)$ is false

  - How many values of (x,y,z) make the predicate true?

    There are infinitely many values that make the proposition true, how many right triangles are there?

Universe of Discourse

- Consider the statement 'x>3', does it make sense to assign to x the value 'blue'?
- Intuitively, the **universe of discourse** is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

EnrolledCSE235(x)='x is enrolled in CSE235'

Universe of Discourse: Multivariate functions

- Each variable in an *n*-tuple (i.e., each argument) may have a different universe of discourse
- Consider an *n*-ary predicate *P*:

  P(*r*,*g*,*b*,*c*)= 'The *rgb*-values of the color *c* is (*r*,*g*,*b*)'
- Example, what is the truth value of
  - *P*(*255,0,0,red*)
  - *P*(*0,0,255,green*)
- What are the universes of discourse of (*r*,*g*,*b*,*c*)?

Alert

- Propositional Logic (PL)
  - Sentential logic
  - Boolean logic
  - Zero order logic
- First Order Logic (FOL)
  - Predicate logic (PL)

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Quantifiers: Introduction

- The statement *‘x>3’* is not a proposition
- It becomes a proposition
  - When we assign values to the argument: ‘4>3’ is true, ‘2<3’ is false, or
  - When we quantify the statement
- Two quantifiers
  - Universal quantifier ∀ $\forall$

    the proposition is true for all possible values in the universe of discourse
  - Existential quantifier ∃ $\exists$

    the proposition is true for some value(s) in the universe of discourse

Universal Quantifier: Definition

- **Definition**: The universal quantification of a predicate $P(x)$ is the proposition '<u>$P(x)$ is true for all values of x in the universe of discourse.</u>' We use the notation: $\forall x\, P(x)$, which is read 'for all x'.
- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements

$$\forall x\, P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \ldots \wedge P(n_k)$$

Universal Quantifier: Example 1

- Let
  - $P(x)$: '*x* must take a discrete mathematics course' and
  - $Q(x)$: '*x* is a CS student.'
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statements:
  - "Every CS student must take a discrete mathematics course."

    $$\forall x\ Q(x) \rightarrow P(x)$$

  - "Everybody must take a discrete mathematics course or be a CS student." $\forall x\ (P(x) \vee Q(x))$
  - "Everybody must take a discrete mathematics course and be a CS student." $\forall x\ (P(x) \wedge Q(x))$

Are these statements true or false at UNL?

Universal Quantifier: Example 2

- Express in FOL the statement

  ‘for every x and every y, $x+y>10$’

- Answer:
  1. Let $P(x,y)$ be the statement $x+y>10$
  2. Where the universe of discourse for x, y is the set of integers
  3. The statement is: $\forall x\ \forall y\ P(x,y)$
- Shorthand: $\forall x,y\ P(x,y)$

Existential Quantifier: Definition

- **Definition**: The existential quantification of a predicate *P*(*x*) is the proposition '<u>There exists a value x in the universe of discourse such that *P*(*x*) is true</u>'
  - Notation: $\exists x\, P(x)$
  - Reads: 'there exists *x*'
- If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the <u>disjunction</u> of the propositions over all the elements

$$\exists x\, P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \ldots \vee P(n_k)$$

Existential Quantifier: Example 1

- Let $P(x,y)$ denote the statement 'x+y=5'
- What does the expression $\exists x \exists y P(x,y)$ mean?
- Which universe(s) of discourse make it true?

Existential Quantifier: Example 2

- Express formally the statement

  'there exists a real solution to $ax^2+bx-c=0$'

- Answer:
  1. Let $P(x)$ be the statement $x= (-b\pm\sqrt{(b^2-4ac)})/2a$
  2. Where the universe of discourse for x is the set of <u>real numbers</u>. Note here that a, b, c are fixed constants.
  3. The statement can be expressed as $\exists x\, P(x)$
- What is the truth value of $\exists x\, P(x)$, where UoD is $\mathbf{R}$?
  – It is false. When $b^2<4ac$, there are no real number x that can satisfy the predicate
- What can we do so that $\exists x\, P(x)$is true?
  – Change the universe of discourse to the complex numbers, $\mathbf{C}$

Quantifiers: Truth values

- In general, when are quantified statements true or false?

Statement	True when...	False when...
$\forall x\, P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\, P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Mixing quantifiers (1)

- Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

$$\forall x \, \exists y \, P(x,y)$$

  is perfectly valid
- Alert:
  - The quantifiers must be read from left to right
  - The order of the quantifiers is important
  - $\forall x \, \exists y \, P(x,y)$ is not equivalent to $\exists y \, \forall x \, P(x,y)$

Mixing quantifiers (2)

- Consider
  - $\forall x\ \exists y\ Loves\ (x,y)$: Everybody loves somebody
  - $\exists y\ \forall x\ Loves(x,y)$: There is someone loved by everyone
- The two expressions do not mean the same thing
- $(\exists y\ \forall x\ Loves(x,y)) \rightarrow (\forall x\ \exists y\ Loves\ (x,y))$

  but the converse does not hold
- However, you can commute similar quantifiers
  - $\forall x\ \forall y\ P(x,y)$ is equivalent to $\forall y\ \forall x\ P(x,y)$ (thus, $\forall x,y\ P(x,y)$)
  - $\exists x\ \exists y\ P(x,y)$ is equivalent to $\exists y\ \exists x\ P(x,y)$ (thus $\exists \mathrm{x},y\ \ P(x,y)$)

Mixing Quantifiers: Truth values

Statement	True when…	False when…
$\forall x \forall y\, P(x,y)$	$P(x,y)$ is true for every pair x,y	There is at least one *pair* x,y for which $P(x,y)$ is false
$\forall x \exists y\, P(x,y)$	For every x, there is a y for which $P(x,y)$ is true	There is an x for which $P(x,y)$ is false for every y
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x, there is a y for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Mixing Quantifiers: Example (1)

- Express, in predicate logic, the statement that there is an infinite number of integers
- Answer:

1. Let $P(x,y)$ be the statement that $x<y$
2. Let the universe of discourse be the integers, Z
3. The statement can be expressed by the following

$$\forall x \, \exists y \, P(x,y)$$

Mixing Quantifiers: Example (2)

- Express the *commutative law of addition* for R
- We want to express that for every pair of reals, x, y, the following holds: $x+y=y+x$
- Answer:
  1. Let $P(x,y)$ be the statement that $x+y$
  2. Let the universe of discourse be the reals, R
  3. The statement can be expressed by the following

$$\forall x\ \forall y\ (P(x,y) \Leftrightarrow P(y,x))$$

Alternatively, $\forall x\ \forall y\ (x+y = y+x)$

Mixing Quantifiers: Example (3)

- Express the multiplicative *law* for nonzero reals $R \setminus \{0\}$ (i.e., every nonzero real has an inverse)
- We want to express that for every real number *x*, there exists a real number *y* such that *xy*=1
- Answer:

$$\forall x \, \exists y \, (xy = 1)$$

Mixing Quantifiers: Example (4)

false mathematical statement

- Does commutativity for substraction hold over the reals?
- That is: does x-y=y-x for all pairs x,y in R?
- Express using quantifiers

$$\forall x \; \forall y \; (x - y = y - x)$$

Mixing Quantifiers: Example (5)

- Express the statement as a logical expression:
  - "There is a number x such that
  - when it is added to any number, the result is that number and
  - if it is multiplied by any number, the result is x"
- Answer:
  - Let $P(x,y)$ be the expression "$x+y=y$"
  - Let $Q(x,y)$ be the expression "$xy=x$"
  - The universe of discourse is N,Z,R,Q *(but not* Z^+*)*
  - Then the expression is:

$$\exists x\ \forall y\ P(x,y) \land Q(x,y)$$

Alternatively: $\exists x\ \forall y\ (x+y=y) \land (xy = x)$

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Binding Variables

- When a quantifier is used on a variable *x*, we say that *x* is <u>bound</u>
- If no quantifier is used on a variable in a predicate statement, the variable is called <u>free</u>
- Examples
  - In $\exists x \forall y P(x,y)$, both *x* and *y* are bound
  - In $\forall x P(x,y)$, *x* is bound but *y* is free
- A statement is called a <u>well-formed formula</u>, when all variables are properly quantified

Binding Variables: Scope

- The set of all variables bound by a common quantifier is called the <u>scope</u> of the quantifier
- For example, in the expression $\exists x,y \forall z P(x,y,z,c)$
  - What is the scope of existential quantifier?
  - What is the scope of universal quantifier?
  - What are the bound variables?
  - What are the free variables?
  - Is the expression a well-formed formula?

Negation

- We can use negation with quantified expressions as we used them with propositions
- **Lemma**: Let $P(x)$ be a predicate. Then the followings hold:

$$\neg (\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg (\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

- This is essentially the quantified version of De Morgan's Law (when the universe of discourse is finite, this is exactly De Morgan's Law)

Negation: Truth

Truth Values of Negated Quantifiers

Statement	True when...	False when...
$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Negation: Example

- Rewrite the following expression, pushing negation inward:

$$\neg\ \forall x\ (\exists\ y\ \forall z\ P(x,y,z) \wedge \exists\ z\ \forall y\ P(x,y,z))$$

- Answer:

$$\exists x\ (\forall y\ \exists z\ \neg P(x,y,z) \vee \forall z\ \exists y\ \neg P(x,y,z))$$

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Prolog (1)

- Prolog (Programming in Logic)
  - is a programming language
  - based on (a restricted form of) Predicate Logic (a.k.a. Predicate Calculus and FOL)
- It was developed
  - by the logicians of the Artificial Intelligence community
  - for symbolic reasoning

Prolog (2)

- Prolog allows the users to express facts and rules
- Facts are propositional functions:
  - student(mia),
  - enrolled(mia,cse235),
  - instructor(patel,cse235), etc.
- Rules are implications with conjunctions:

  teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:

  ?enrolled(mia,cse235)

  ?enrolled(X,cse476)

  ?teaches(X,mia)

  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5

English into Logic

- Logic is more precise than English
- Transcribing English into Logic and vice versa can be tricky
- When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

Use $\forall$ with $\Rightarrow$

$\forall x\ Lion(x) \Rightarrow Fierce(x)$: Every lion is fierce

$\forall x\ Lion(x) \land Fierce(x)$: Everyone is a lion and everyone is fierce

Use $\exists$ with $\land$

$\exists\ x\ Lion(x) \land Vegan(x)$: Holds when you have at least one vegan lion

$\exists\ x\ Lion(x) \Rightarrow Vegan(x)$: Holds when you have vegan people in the universe of discourse (even though there is no vegan lion in the universe of discourse)

More Exercises (1)

- Let P(x,y) denote 'x is a factor of y' where
  - $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$
- Let $Q(x,y)$ denote:
  - $\forall x,y\ \ [P(x,y) \rightarrow (x=y) \vee (x=1)]$
- Question: When is $Q(x,y)$ true?

Alert…

- Some students wonder if:

$$\forall x,y\ P(x,y) \equiv (\forall x\ P(x,y)) \wedge (\forall y\ P(x,y))$$

- This is certainly not true.
  - In the left-hand side, both x,y are bound.
  - In the right-hand side,
    - In the first predicate, *x* is bound and *y* is free
    - In the second predicate, *y* is bound and *x* is free
    - Thus, the left-hand side is a proposition, but the right-hand side is not. They cannot be equivalent
- All variables that occur in a propositional function must be bound to turn it into a proposition