Combinatorics

Section 6.1—6.6 8.5—8.6 of Rosen

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CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Questions: Piazza

Motivation

- Combinatorics is the study of collections of objects. Specifically, counting objects, arrangement, derangement, etc. along with their mathematical properties
- Counting objects is important in order to analyze algorithms and compute discrete probabilities
- Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability
  - A simple example: How many arrangements are there of a deck of 52 cards?
- In addition, combinatorics can be used as a proof technique
  - A combinatorial proof is a proof method that uses counting arguments to prove a statement

Outline

- **Introduction**
- **Counting:**
  - **Product rule, sum rule, Principal of Inclusion Exclusion (PIE)**
  - **Application of PIE: Number of onto functions**
- Pigeonhole principle
  - Generalized, probabilistic forms
- Permutations
- Combinations
- Binomial Coefficients
- Generalizations
  - Combinations with repetitions, permutations with indistinguishable objects
- Algorithms
  - Generating combinations (1), permutations (2)
- More Examples

Product Rule

- If two events are <u>not</u> mutually exclusive (that is we do them separately), then we apply the product rule
- **Theorem**: Product Rule

  Suppose a procedure can be accomplished with two <u>disjoint</u> subtasks. If there are

  – n_1 ways of doing the first task and

  – n_2 ways of doing the second task,

  then there are $n_1.n_2$ ways of doing the overall procedure

Sum Rule (1)

- If two events <u>are</u> mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule
- **Theorem**: Sum Rule. If
  - an event e_1 can be done in n_1 ways,
  - an event e_2 can be done in n_2 ways, and
  - e_1 and e_2 are mutually exclusive

  then the number of ways of both events occurring is $n_1 + n_2$

Sum Rule (2)

- There is a natural generalization to any <u>sequence</u> of m tasks; namely the number of ways m mutually exclusive events can occur

$$n_1 + n_2 + \ldots + n_{m-1} + n_m$$

- We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be <u>pairwise disjoint sets.</u> Then

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| \cup |A_2| \cup \ldots \cup |A_m|$$

(In fact, this is a special case of the general Principal of Inclusion-Exclusion (PIE))

Principle of Inclusion-Exclusion (PIE)

- Say there are two events, e_1 and e_2, for which there are n_1 and n_2 possible outcomes respectively.
- Now, say that only <u>one</u> event can occur, not both
- In this situation, we cannot apply the sum rule. Why?

… because we would be over counting the number of possible outcomes.

- Instead we have to count the number of possible outcomes of e_1 and e_2 minus the number of possible outcomes in common to both; i.e., the number of ways to do both tasks
- If again we think of them as sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

PIE (2)

- More generally, we have the following
- **Lemma**: Let A, B, be subsets of a finite set U. Then
  1. $|A \cup B| = |A| + |B| - |A \cap B|$
  2. $|A \cap B| \leq \min\{|A|, |B|\}$
  3. $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
  4. $|\overline{A}| = |U| - |A|$
  5. $|A \oplus B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|$
  6. $|A \times B| = |A| \times |B|$

PIE: Theorem

- **Theorem**: Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = & \Sigma_i |A_i| \\
& - \Sigma_{i<j} |A_i \cap A_j| \\
& + \Sigma_{i<j<k} |A_i \cap A_j \cap A_k| \\
& - \ldots \\
& + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|
\end{aligned}
$$

Each summation is over

- all i,
- pairs i,j with i<j,
- triples with i<j<k, etc.

PIE Theorem: Example 1

- To illustrate, when n=3, we have

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3| = {} & |A_1| + |A_2| + |A_3| \\
& - [|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|] \\
& + |A_1 \cap A_2 \cap A_3|
\end{aligned}
$$

PIE Theorem: Example 2

- To illustrate, when n=4, we have

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - [|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\
& \quad + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|] \\
& + [|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
& \quad + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|] \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}
$$

Application of PIE: Example A (1)

- How many integers between 1 and 300 (inclusive) are
  - Divisible by at least one of 3,5,7?
  - Divisible by 3 and by 5 but not by 7?
  - Divisible by 5 but by neither 3 or 7?
- Let

  $A = \{n \in Z \mid (1 \leq n \leq 300) \wedge (3 \mid n)\}$

  $B = \{n \in Z \mid (1 \leq n \leq 300) \wedge (5 \mid n)\}$

  $C = \{n \in Z \mid (1 \leq n \leq 300) \wedge (7 \mid n)\}$
- How big are these sets? We use the floor function

  $|A| = \lfloor 300/3 \rfloor = 100$

  $|B| = \lfloor 300/5 \rfloor = 60$

  $|C| = \lfloor 300/7 \rfloor = 42$

Application of PIE: Example A (2)

- How many integers between 1 and 300 (inclusive) are divisible by at least one of 3,5,7?

  Answer: $|A \cup B \cup C|$
- By the principle of inclusion-exclusion

  $$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$$
- How big are these sets? We use the floor function

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad$ $|A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad$ $|A \cap C| = \lfloor 300/21 \rfloor = 100$

  $|C| = \lfloor 300/7 \rfloor = 42$ $\quad$ $|B \cap C| = \lfloor 300/35 \rfloor = 8$

  $|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$
- Therefore:

  $$|A \cup B \cup C| = 100 + 60 + 42 - (20+14+8) + 2 = 162$$

Application of PIE: Example A (3)

- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?

  Answer: $|(A \cap B)\backslash C|$
- By the definition of set-minus

$$|(A \cap B)\backslash C| = |A \cap B| - |A \cap B \cap C| = 20 – 2 = 18$$

- Knowing that

$\lvert A\rvert = \lfloor 300/3 \rfloor = 100$	$\lvert A\cap B\rvert = \lfloor 300/15 \rfloor = 20$
$\lvert B\rvert = \lfloor 300/5 \rfloor = 60$	$\lvert A\cap C\rvert = \lfloor 300/21 \rfloor = 100$
$\lvert C\rvert = \lfloor 300/7 \rfloor = 42$	$\lvert B\cap C\rvert = \lfloor 300/35 \rfloor = 8$
	$\lvert A\cap B\cap C\rvert = \lfloor 300/105 \rfloor = 2$

Application of PIE: Example A (4)

- How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?

  Answer: $|B \backslash (A \cup C)| = |B| - |B \cap (A \cup C)|$
- Distributing B over the intersection

  $$|B \cap (A \cup C)| = |(B \cap A) \cup (B \cap C)|$$
  $$= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|$$
  $$= |B \cap A| + |B \cap C| - |B \cap A \cap C|$$
  $$= 20 + 8 – 2 = 26$$

- Knowing that

  $|A| = \lfloor 300/3 \rfloor = 100$ $\quad$ $|A \cap B| = \lfloor 300/15 \rfloor = 20$

  $|B| = \lfloor 300/5 \rfloor = 60$ $\quad$ $|A \cap C| = \lfloor 300/21 \rfloor = 14$

  $|C| = \lfloor 300/7 \rfloor = 42$ $\quad$ $|B \cap C| = \lfloor 300/35 \rfloor = 8$

  $|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$

Application of PIE: #Surjections (Section 7.6)

- The principle of inclusion-exclusion can be used to count the number of onto (surjective) functions
- **Theorem**: Let A, B be non-empty sets of cardinality m,n with m≥n. Then there are

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

i.e. $\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m$ *onto functions* $f : A \to B$.

${n \choose i}$

See textbook, Section 8.6 page 561

#Surjections: Example

- How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?
- This problem can be modeled as follows:
  - Let A be the set of candies, |A|=6
  - Let B be the set of children, |B|=3
  - The problem becomes "find the number of surjective mappings from A to B" (because each child must receive at least one candy)
- Thus the number of ways is thus (m=6, n=3)

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$

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Pigeonhole Principle (1)

- If there are more pigeons than there are roots (pigeonholes), for at least one pigeonhole, more than one pigeon must be in it
- **Theorem**: If k+1 or more objects are placed in k boxes, then there is at least one box containing two or more objects
- This principal is a fundamental tool of elementary discrete mathematics.
- It is also known as the <u>Dirichlet Drawer Principle</u> or <u>Dirichlet Box Pinciple</u>

Pigeonhole Principle (2)

- It is <u>seemingly</u> simple but <u>very</u> powerful
- The difficulty comes in where and how to apply it
- Some simple applications in Computer Science
  - Calculating the probability of hash functions having a collision
  - Proving that there can be no lossless compression algorithm compressing all files to within a certain ration
- **Lemma**: For two finite sets A,B there exists a bijection $f:A \to B$ if and only if $|A|=|B|$

Generalized Pigeonhole Principle (1)

- **Theorem**: If N objects are placed into k boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

- **Example**: In any group of 367 or more people, at least two of them must have been born on the same date.

Generalized Pigeonhole Principle (2)

- A probabilistic generalization states that
  - if n objects are randomly put into m boxes
  - with uniform probability
  - (i.e., each object is place in a given box with probability 1/m)
  - then at least one box will hold more than one object with probability

$$1 - \frac{m!}{(m-n)!m^n}$$

Generalized Pigeonhole Principle: Example

- Among 10 people, what is the probability that two or more will have the same birthday?
  - Here n=10 and m=365 (ignoring leap years)
  - Thus, the probability that two will have the same birthday is

$$1 - \frac{365!}{(365-10)!365^{1}0} \approx 0.1169$$

So, less than 12% probability

Pigeonhole Principle: Example A (1)

- Show that
  - in a room of n people with certain acquaintances,
  - some pair must have the same number of acquaintances
- Note that this is equivalent to showing that any symmetric, irreflexive relation on n elements must have two elements with the same number of relations
- Proof: by contradiction using the pigeonhole principle
- Assume, to the contrary, that every person has a different number of acquaintances: 0, 1, 2, …, n-1 (no one can have n acquaintances because the relation is irreflexive).
- There are n possibilities, we have n people, we are not done ☹

Pigeonhole Principle: Example A (2)

- Assume, to the contrary, that every person has a different number of acquaintances: 0, 1, 2, …, n-1 (no one can have n acquaintances because the relation is irreflexive).
- There are n possibilities, we have n people, we are not done ☹
- We need to use the fact that acquaintanceship is a symmetric irreflexive relation
- In particular, some person knows 0 people while another knows n-1 people
- This is impossible. Contradiction! ☺ So we do not have n (10) possibilities, but less
- Thus by the pigeonhole principle (10 people and 9 possibilities) at least two people have to the same number of acquaintances

Pigeonhole Principle: Example B

- **Example**: Say, 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats.
- **Show that**
  - One of the buses will have 14 empty seats
  - One of the buses will carry at least 67 passengers
- *One of the buses will have 14 empty seats*
  - Total number of seats is 80.30=2400
  - Total number of empty seats is 2400-2000=400
  - By the pigeonhole principle: 400 empty seats in 30 buses, one must have $\lceil 400/30 \rceil = 14$ empty seats
- *One of the buses will carry at least 67 passengers*
  - By the pigeonhole principle: 2000 passengers in 30 buses, one must have $\lceil 2000/30 \rceil = 67$ passengers

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Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set of n elements is called an r-permutation
- **Theorem**: The number of r permutations of a set of n distinct elements is
$$P(n,r) = \Pi_{i=0}^{r-1}(n-i) = n(n-1)(n-2)\cdots(n-r+1)$$
- It follows that $$P(n,r) = \frac{n!}{(n-r)!}$$
- In particular $$P(n,n) = n!$$
- Note here that the order is important. It is necessary to distinguish when the order matters and it does not

Application of PIE and Permutations: Derangements (I) (Section 7.6)

- Consider the hat-check problem
  - Given
    - An employee checks hats from n customers
    - However, s/he forgets to tag them
    - When customers check out their hats, they are given one at random
  - Question
    - What is the probability that no one will get their hat back?

Application of PIE and Permutations: Derangements (II)

- The hat-check problem can be modeled using derangements: permutations of objects such that no element is in its original position

  - Example: 21453 is a derangement of 12345 but 21543 is not
- The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!}\right]$$

- Thus, the answer to the hatcheck problem is $\frac{D_n}{n!}$
- Note that $$e^{-1} = \left[1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!}\right]$$
- Thus, the probability of the hatcheck problem converges

$$\lim_{n \to \infty} \frac{D_n}{n!} = e^{-1} \approx 0.368$$

See textbook, Section 8.6 page 562

Permutations: Example A

- How many pairs of dance partners can be selected from a group of 12 women and 20 men?
  - The first woman can partner with any of the 20 men, the second with any of the remaining 19, etc.
  - To partner all 12 women, we have

$$P(20,12) = 20!/8! = 9.10.11\ldots20$$

Permutations: Example B

- In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?
  - The number of ways is P(24,10)
  - Since we can choose either a or z to come first, then there are 2P(24,10) arrangements of the 12-letter block
  - For the remaining 14 letters, there are P(15,15)=15! possible arrangements
  - In all there are 2P(24,10).15! arrangements

Permutations: Example C (1)

- How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern *bge* nor *eaf*?
  - The total number of permutations is P(7,7)=7!
  - If we fix the pattern *bge*, then we consider it as a single block. Thus, the number of permutations with this pattern is P(5,5)=5!
  - If we fix the pattern *bge*, then we consider it as a single block. Thus, the number of permutations with this pattern is P(5,5)=5!
  - Fixing the patter eaf, we have the same number: 5!
  - Thus, we have (7! – 2.5!). Is this correct?
  - No! we have subtracted too many permutations: ones containing both *eaf* and *bfe*.

Permutations: Example C (2)

- There are two cases: (1) *eaf* comes first, (2) *bge* comes first
- Are there any cases where *eaf* comes before *bge*?
- No! The letter e cannot be used twice
- If *bge* comes first, then the pattern must be *bgeaf*, so we have 3 blocks or 3! arrangements
- Altogether, we have

$$7! - 2.(5!) + 3! = 4806$$

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Combinations (1)

- Whereas permutations consider order, combinations are used when order does not matter
- **Definition**: A k-combination of elements of a set is an unordered selection of k elements from the set.

  (A combination is imply a subset of cardinality k)

Combinations (2)

- **Theorem**: The number of k-combinations of a set of cardinality n with $0 \leq k \leq n$ is

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

is read 'n choose k'. ${n \choose k}$

Combinations (3)

- A useful fact about combinations is that they are symmetric

$$\binom{n}{1} = \binom{n}{n-1} \quad \binom{n}{2} = \binom{n}{n-2} \quad \binom{n}{3} = \binom{n}{n-3}$$

- **Corollary**: Let n, k be nonnegative integers with k ≤ n, then

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinations: Example A

- In the Powerball lottery, you pick
  - five numbers between 1 and 55 and
  - A single 'powerball' number between 1 and 42

  How many possible plays are there?
- Here order does not matter
  - The number of ways of choosing 5 numbers is $\binom{55}{5}$
  - There are 42 possible ways to choose the powerball
  - The two events are not mutually exclusive: $42\binom{55}{5}$
  - The odds of winning are $\frac{1}{42\binom{55}{5}} < 0.000000006845$

Combinations: Example B

- In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?
  - The number of ways of choosing 3 heads out of 10 coin tosses is $\binom{10}{3}$
  - It is the same as choosing 7 tails out of 10 coin tosses $\binom{10}{7} = \binom{10}{3} = 120$
  - … which illustrates the corollary $\binom{n}{k} = \binom{n}{n-k}$

Combinations: Example C

- How many committees of 5 people can be chosen from 20 men and 12 women
  - If exactly 3men must be on each committee?
  - If at least 4 women must be on each committee?
- *If exactly three men must be on each committee?*
  - We must choose 3 men and 2 women. The choices are <u>not</u> mutually exclusive, we use the product rule

$$\binom{20}{3} \cdot \binom{12}{2}$$

- *If at least 4 women must be on each committee?*
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. Theses choices are mutually exclusive, we use the addition rule:

$$\binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692$$

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Binomial Coefficients (1)

- The number of r-combinations $\binom{n}{r}$ is also called the <u>binomial coefficient</u>

- The binomial coefficients are the coefficients in the expansion of the expression, (multivariate polynomial),

$$(x+y)^n$$

- A binomial is a sum of two terms

Binomial Coefficients (2)

- **Theorem**: Binomial Theorem

  Let x, y, be variables and let n be a nonnegative integer. Then

$$(x+y)^n = \Sigma_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

  Expanding the summation we have

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

  Example $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Binomial Coefficients: Example

- What is the coefficient of the term x^8y^{12} in the expansion of $(3x+4y)^{20}$?
  - By the binomial theorem, we have

$$(3x+4y)^{20} = \Sigma_{j=0}^{20} \binom{20}{j} (3x)^{n-j} (4y)^j$$

  - When j=12, we have

$$\binom{20}{12} (3x)^8 (4y)^{12}$$

  - The coefficient is

$$\binom{20}{12} 3^8 4^{12} = \frac{20!}{12!8!} 3^8 4^{12} = 13866187326750720$$

Binomial Coefficients (3)

- Many useful identities and facts come from the Binomial Theorem
- **Corollary**:

$$\Sigma_{k=0}^{n}\binom{n}{k} = 2^n$$

$$\Sigma_{k=0}^{n}(-1)^k\binom{n}{k} = 0, \qquad n \geq 1$$

$$\Sigma_{k=0}^{n}2^k\binom{n}{k} = 3^n$$

Equalities are based on $(1+1)^n=2^n$, $((-1)+1)^n=0^n$, $(1+2)^n=3^n$

Binomial Coefficients (4)

- **Theorem**: Vandermonde's Identity

  Let m,n,r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \Sigma_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- **Corollary**: If n is a nonnegative integer then $\binom{2n}{n} = \Sigma_{k=0}^{r} \binom{n}{k}^2$

- **Corollary**: Let n,r be nonnegative integers, r≤n, then

$$\binom{n+1}{r+1} = \Sigma_{j=r}^{n} \binom{j}{r}$$

Binomial Coefficients: Pascal's Identity & Triangle

- The following is known as Pascal's identity which gives a useful identity for efficiently computing binomial coefficients
- **Theorem**: Pascal's Identity

  Let n,k $\in Z^{+}$ with n≥k, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Identity forms the basis of a geometric object known as Pascal's Triangle

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

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Generalized Combinations & Permutations (1)

- Sometimes, we are interested in permutations and combinations in which repetitions are allowed
- **Theorem**: The number of r-permutations of a set of n objects with repetition allowed is n^r

…which is easily obtained by the product rule

- **Theorem**: There are

$$\binom{n+r-1}{r}$$

r-combinations from a set with n elements when repetition of elements is allowed

Generalized Combinations & Permutations: Example

- There are 30 varieties of donuts from which we wish to buy a dozen. How many possible ways to place your order are there?
- Here, n=30 and we wish to choose r=12.
- Order does not matter and repetitions are possible
- We apply the previous theorem
- The number of possible orders is

$$\binom{n+r-1}{r} = \binom{30+12-1}{12} = \binom{17}{12}$$

Generalized Combinations & Permutations (2)

- **Theorem:** The number of different <u>permutations</u> of n objects where there are n_1 indistinguishable objects of type 1, n_2 of type 2, and n_k of type k is

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

An equivalent ways of interpreting this theorem is the number of ways to

  - distribute n distinguishable objects
  - into k distinguishable boxes
  - so that n_i objects are place into box i for i=1,2,3,…,k

Example

- How many permutations of the word Mississipi are there?
- 'Mississipi' has
  - 4 distinct letters: m,i,s,p
  - with 1,4,4,2 occurrences respectively
  - Therefore, the number of permutations is

$$\frac{11!}{1!4!4!2!}$$

Outline

- Introduction
- Counting:
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  - Application of PIE: Number of onto functions
- Pigeonhole principle
  - Generalized, probabilistic forms
- Permutations
- Combinations
- Binomial Coefficients
- Generalizations
  - Combinations with repetitions, permutations with indistinguishable objects
- **Algorithms**
  - **Generating combinations (1), permutations (2)**
- More Examples

Algorithms

- In general, it is inefficient to solve a problem by considering all permutation or combinations since there are exponential (worst, factorial!) numbers of such arrangements
- Nevertheless, for many problems, no better approach is known.
- When exact solutions are needed, backtracking algorithms are used to exhaustively enumerate all arrangements

Algorithms: Example

- **Traveling Salesperson Problem** (TSP)

  Consider a salesman that must visit n different cities. He wishes to visit them in an order such that his overall distance travelled is minimized

- This problem is one of hundred of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that no efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP.)
- The only way of solving this problem <u>exactly</u> is to try all possible n! routes
- We give several algorithms for generating these combinatorial objects

Generating Combinations (1)

- Recall that combinations are simply all possible subsets of size r. For our purposes, we will consider generating subsets of

$$\{1,2,3,\ldots,n\}$$

- The algorithm works as follows
  - Start with $\{1,\ldots,r\}$
  - Assume that we have $a_1a_2\ldots a_r$, we want the next combination
  - Locate the last element ai such that $a_i \neq n\text{-}r\text{-}I$
  - Replace a_i with a_i+1
  - Replace a_j with $a_i+j\text{-}I$ for $j=i+1, i+2,\ldots,r$

Generating Combinations (2)

Next r-Combinations

Input: A set of n elements and an r-combination $a_1, a_2, \ldots, a_r$

Output: The next r-combination

1. $i \leftarrow r$
2. **While** $a_i = n-r+i$ **Do**
3. $\quad i \leftarrow i-1$
4. **End**
5. $a_i \leftarrow a_i + 1$
6. **For** $j \leftarrow (i+1)$ **to** r **Do**
7. $\quad a_j \leftarrow a_i + j - i$
8. **End**

Generating Combinations: Example

- Find the next 3-combination of the set {1,2,3,4,5} after {1,4,5}
- Here $a_1=1$, $a_2=4$, $a_3=5$, $n=5$, $r=3$
- The last i such that $a_i \neq 5-3+i$ is 1
- Thus, we set

  $a_1 = a_1 + 1 = 2$

  $a_2 = a_1 + 2 - 1 = 3$

  $a_3 = a_1 + 3 - 1 = 4$

  Thus, the next r-combinations is {2,3,4}

Generating Permutations

- The textbook gives an algorithm to generate permutations in lexicographic order. Essentially, the algorithm works as follows. Given a permutation
  - Choose the left-most pair a_j,a_{j+1} where a_j<a_{j+1}
  - Choose the least items to the right of a_j greater than a_j
  - Swap this item and a_j
  - Arrange the remaining (to the right) items in order

Next Permutation (lexicographic order)

Input : *A set of n elements and an r-permutation, $a_1 \cdots a_r$.*

Output : *The next r-permutation.*

$j = n - 1$
WHILE $a_j > a_{j+1}$ DO
    $j = j - 1$
END
//j is the largest subscript with $a_j < a_{j+1}$
$k = n$
WHILE $a_j > a_k$ DO
    $k = k - 1$
END
//a_k is the smallest integer greater than a_j to the right of a_j
swap(a_j, a_k)
$r = n$
$s = j + 1$
WHILE $r > s$ DO
    swap(a_r, a_s)
    $r = r - 1$
    $s = s + 1$
END

Generating Permutations (2)

- Often there is no reason to generate permutations in lexicographic order. Moreover even though generating permutations is inefficient in itself, lexicographic order induces even more work
- An alternate method is to fix an element, then recursively permute the n-1 remaining elements
- The Johnson-Trotter algorithm has the following attractive properties. Not in your textbook, not on the exam, just for your reference/culture
  - It is bottom up (non-recursive)
  - It induces a minimal-change between each permutation

Johnson-Trotter Algorithm

- We associate a direction to each element, for example

$$\overrightarrow{3}\ \overleftarrow{2}\ \overrightarrow{4}\ \overleftarrow{1}$$

- A component is mobile if its direction points to an adjacent component that is smaller than itself.
- Here 3 and 4 are mobile, 1 and 2 are not

Algorithm: Johnson Trotter

INPUT : *An integer* n.

OUTPUT : *All possible permutations of* $\langle 1, 2, \ldots n \rangle$.

$\pi = \overleftarrow{1}\,\overleftarrow{2} \ldots \overleftarrow{n}$

WHILE *There exists a mobile integer* $k \in \pi$ DO

  $k =$ *largest mobile integer*

  swap k *and the adjacent integer* k *points to*

  reverse direction of all integers $> k$

  Output π

END

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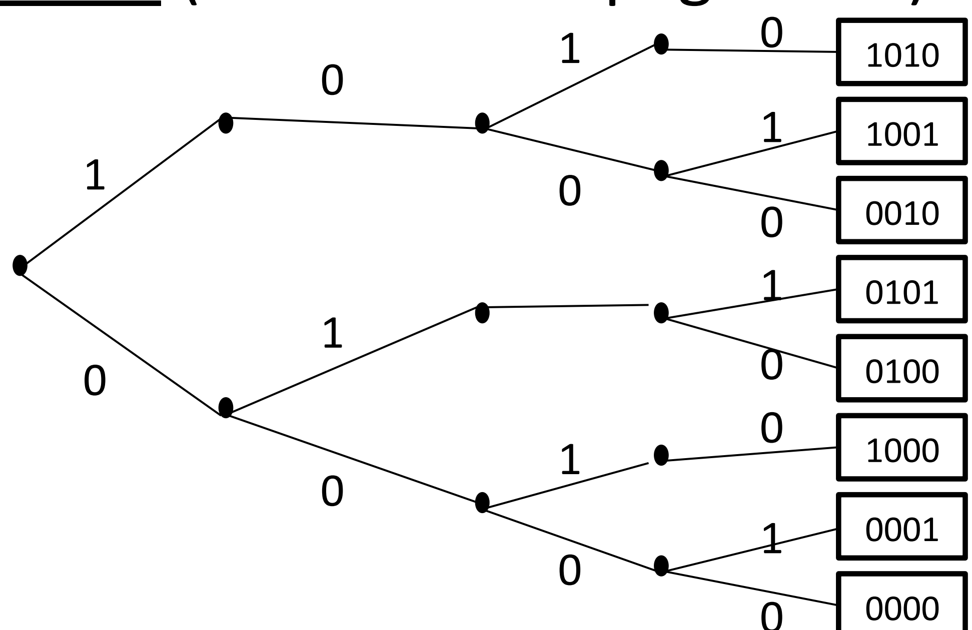
Example A

- How many bit strings of length 4 are there such that 11 never appear as a substring
- We can represent the set of strings graphically using a <u>diagram tree</u> (see textbook pages 395)



Example: Counting Functions (1)

- Let S,T be sets such that |S|=n, |T|=m.
  - How many function are there mapping $f:S \rightarrow T$?
  - How many of these functions are one-to-one (injective)?
- A function simply maps each s_i to one t_j, thus for each n we can choose to send it to any of the elements in T
- Each of these is an independent event, so we apply the multiplication rule:
- If we wish f to be injective, we must have $n \leq m$, otherwise the answer is obviously 0

Example: Counting Functions (2)

- Now each s_i must be mapped to a unique element in T.
  - For s_1, we have m choices
  - However, once we have made a mapping, say s_j, we cannot map subsequent elements to t_j again
  - In particular, for the second element, s_2, we now have m-1 choices, for s_3, m-2 choices, etc.

$$m \cdot (m-1) \cdot (m-2) \cdot \ldots (m-(n-2)) \cdot (m-(n-1))$$

- An alternative way of thinking is using the choose operator: we need to choose n element from a set of size m for our mapping

$$\binom{m}{n} = \frac{m!}{(m-n)!n!}$$

- Once we have chosen this set, we now consider all permutations of the mapping, that is n! different mappings for this set. Thus, the number of such mapping is $\frac{m!}{(m-n)!n!} \cdot n! = \frac{m!}{(m-n)!}$

Another Example: Counting Functions

- Let S={1,2,3}, T={a,b}.
  - How many onto (surjective) mappings are there from S→T?
  - How many onto-to-one injective functionsare there from T→S?
- See Theorem 1, page 561

Example: Sets

- How many k integers $1 \leq k \leq 100$ are divisible by 2 or 3?
- Let
  - $A = \{n \in Z \mid (1 \leq n \leq 100) \wedge (2|n)\}$
  - $B = \{n \in Z \mid (1 \leq n \leq 100) \wedge (3|n)\}$
- Clearly, $|A| = \lfloor 100/2 \rfloor = 50$, $|B| = \lfloor 100/3 \rfloor = 33$
- Do we have $|A \cup B| = 83$? No!
- We have over counted the integers divisible by 6
  - Let $C = \{n \in Z \mid (1 \leq n \leq 100) \wedge (6|n)\}$, $|C| = \lfloor 100/6 \rfloor = 16$
- So $|A \cup B| = (50+33) - 16 = 67$

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