Combinatorics

Section 6.1—6.6  8.5—8.6 of Rosen
Spring 2012
CSCE 235 Introduction to Discrete Structures
Course web-page: cse.unl.edu/~cse235
Questions: Piazza
Motivation

• Combinatorics is the study of collections of objects. Specifically, counting objects, arrangement, derangement, etc. along with their mathematical properties

• Counting objects is important in order to analyze algorithms and compute discrete probabilities

• Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability
  – A simple example: How many arrangements are there of a deck of 52 cards?

• In addition, combinatorics can be used as a proof technique
  – A combinatorial proof is a proof method that uses counting arguments to prove a statement
Outline

• Introduction

• Counting:
  – Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  – Application of PIE: Number of onto functions

• Pigeonhole principle
  – Generalized, probabilistic forms

• Permutations

• Combinations

• Binomial Coefficients

• Generalizations
  – Combinations with repetitions, permutations with indistinguishable objects

• Algorithms
  – Generating combinations (1), permutations (2)

• More Examples
Product Rule

• If two events are not mutually exclusive (that is we do them separately), then we apply the product rule

• **Theorem**: Product Rule

Suppose a procedure can be accomplished with two disjoint subtasks. If there are

– $n_1$ ways of doing the first task and
– $n_2$ ways of doing the second task,

then there are $n_1 \cdot n_2$ ways of doing the overall procedure
Sum Rule (1)

• If two events are mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule

• **Theorem**: Sum Rule. If
  – an event $e_1$ can be done in $n_1$ ways,
  – an event $e_2$ can be done in $n_2$ ways, and
  – $e_1$ and $e_2$ are mutually exclusive
then the number of ways of both events occurring is $n_1 + n_2$
Sum Rule (2)

There is a natural generalization to any sequence of m tasks; namely the number of ways m mutually exclusive events can occur

\[ n_1 + n_2 + \ldots + n_{m-1} + n_m \]

We can give another formulation in terms of sets. Let \( A_1, A_2, \ldots, A_m \) be pairwise disjoint sets. Then

\[ |A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| \cup |A_2| \cup \ldots \cup |A_m| \]

(In fact, this is a special case of the general Principal of Inclusion-Exclusion (PIE))
Principle of Inclusion-Exclusion (PIE)

• Say there are two events, $e_1$ and $e_2$, for which there are $n_1$ and $n_2$ possible outcomes respectively.

• Now, say that only one event can occur, not both.

• In this situation, we cannot apply the sum rule. Why?
  
  ... because we would be over counting the number of possible outcomes.

• Instead we have to count the number of possible outcomes of $e_1$ and $e_2$ minus the number of possible outcomes in common to both; i.e., the number of ways to do both tasks.

• If again we think of them as sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
PIE (2)

More generally, we have the following

**Lemma:** Let \( A, B, \) be subsets of a finite set \( U. \) Then

1. \(|A \cup B| = |A| + |B| - |A \cap B|\)
2. \(|A \cap B| \leq \min \{|A|, |B|\}\)
3. \(|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|\)
4. \(|A| = |U| - |A|\)
5. \(|A \oplus B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|\)
6. \(|A \times B| = |A| \times |B|\)
**PIE: Theorem**

- **Theorem**: Let $A_1, A_2, \ldots, A_n$ be finite sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i|$$

- $- \sum_{i < j} |A_i \cap A_j|$ 

+ $\sum_{i < j < k} |A_i \cap A_j \cap A_k|$ 

- ... 

$+(-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$ 

Each summation is over

- all $i$, 
- pairs $i,j$ with $i < j$, 
- triples with $i < j < k$, etc.
PIE Theorem: Example 1

• To illustrate, when n=3, we have

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

$$- [ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| ]$$

$$+ |A_1 \cap A_2 \cap A_3|$$
PIE Theorem: Example 2

• To illustrate, when n=4, we have

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| \]

\[ - [ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| ] \]

\[ + [ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| ] \]

\[ - |A_1 \cap A_2 \cap A_3 \cap A_4| \]
Application of PIE: Example A (1)

• How many integers between 1 and 300 (inclusive) are
  – Divisible by at least one of 3, 5, 7?
  – Divisible by 3 and by 5 but not by 7?
  – Divisible by 5 but by neither 3 or 7?

• Let
  \[ A = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (3 \mid n) \} \]
  \[ B = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (5 \mid n) \} \]
  \[ C = \{ n \in \mathbb{Z} \mid (1 \leq n \leq 300) \land (7 \mid n) \} \]

• How big are these sets? We use the floor function
  \[ |A| = \left\lfloor \frac{300}{3} \right\rfloor = 100 \]
  \[ |B| = \left\lfloor \frac{300}{5} \right\rfloor = 60 \]
  \[ |C| = \left\lfloor \frac{300}{7} \right\rfloor = 42 \]
Application of PIE: Example A (2)

• How many integers between 1 and 300 (inclusive) are divisible by at least one of 3, 5, 7?

Answer: $|A \cup B \cup C|

• By the principle of inclusion-exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

• How big are these sets? We use the floor function

$|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100$  
$|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20$

$|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60$  
$|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 100$

$|C| = \left\lfloor \frac{300}{7} \right\rfloor = 42$  
$|B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8$

$|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2$

• Therefore:

$$|A \cup B \cup C| = 100 + 60 + 42 - (20+14+8) + 2 = 162$$
Application of PIE: Example A (3)

- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?
  Answer: |(A ∩ B)\C|

- By the definition of set-minus
  \[|(A \cap B)\setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18\]

- Knowing that
  \[|A| = \left\lceil \frac{300}{3} \right\rceil = 100 \quad |A\cap B| = \left\lceil \frac{300}{15} \right\rceil = 20\]
  \[|B| = \left\lceil \frac{300}{5} \right\rceil = 60 \quad |A\cap C| = \left\lceil \frac{300}{21} \right\rceil = 100\]
  \[|C| = \left\lceil \frac{300}{7} \right\rceil = 42 \quad |B\cap C| = \left\lceil \frac{300}{35} \right\rceil = 8\]
  \[|A\cap B\cap C| = \left\lceil \frac{300}{105} \right\rceil = 2\]
Application of PIE: Example A (4)

• How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?
  
  Answer: \(|B \setminus (A \cup C)| = |B| - |B \cap (A \cup C)|

• Distributing \( B \) over the intersection
  
  \(|B \cap (A \cup C)| = |(B \cap A) \cup (B \cap C)|
  
  = |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)|
  
  = |B \cap A| + |B \cap C| - |B \cap A \cap C|
  
  = 20 + 8 - 2 = 26

• Knowing that
  
  \(|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100\)
  \(|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20\)
  \(|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60\)
  \(|A \cap C| = \left\lfloor \frac{300}{21} \right\rfloor = 14\)
  \(|C| = \left\lfloor \frac{300}{7} \right\rfloor = 42\)
  \(|B \cap C| = \left\lfloor \frac{300}{35} \right\rfloor = 8\)
  \(|A \cap B \cap C| = \left\lfloor \frac{300}{105} \right\rfloor = 2\)
Application of PIE: #Surjections
(Section 7.6)

• The principle of inclusion-exclusion can be used to count the number of onto (surjective) functions

• **Theorem**: Let $A, B$ be non-empty sets of cardinality $m, n$ with $m \geq n$. Then there are

\[
\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n - i)^m \] 

onto functions $f : A \rightarrow B$.

$\binom{n}{i}$

See textbook, Section 8.6 page 561
# Surjections: Example

• How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?

• This problem can be modeled as follows:
  – Let A be the set of candies, \(|A| = 6\)
  – Let B be the set of children, \(|B| = 3\)
  – The problem becomes “find the number of surjective mappings from A to B” (because each child must receive at least one candy)

• Thus the number of ways is thus \((m=6, n=3)\)

\[
3^6 - \binom{3}{1}(3 - 1)^6 + \binom{3}{2}(3 - 2)^6 = 540
\]
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Pigeonhole Principle (1)

• If there are more pigeons than there are roots (pigeonholes), for at least one pigeonhole, more than one pigeon must be in it

• **Theorem**: If $k+1$ or more objects are placed in $k$ boxes, then there is at least one box containing two or more objects

• This principal is a fundamental tool of elementary discrete mathematics.

• It is also known as the **Dirichlet Drawer Principle** or **Dirichlet Box Principle**
Pigeonhole Principle (2)

• It is **seemingly** simple but **very** powerful
• The difficulty comes in where and how to apply it
• Some simple applications in Computer Science
  – Calculating the probability of hash functions having a collision
  – Proving that there can be no lossless compression algorithm compressing all files to within a certain ration

• **Lemma**: For two finite sets A,B there exists a bijection f:A→B if and only if |A|=|B|
Generalized Pigeonhole Principle (1)

• **Theorem:** If \( N \) objects are placed into \( k \) boxes then there is at least one box containing at least

\[
\left\lfloor \frac{N}{k} \right\rfloor
\]

• **Example:** In any group of 367 or more people, at least two of them must have been born on the same date.
Generalized Pigeonhole Principle (2)

• A probabilistic generalization states that
  – if $n$ objects are randomly put into $m$ boxes
  – with uniform probability
  – (i.e., each object is place in a given box with probability $1/m$)
  – then at least one box will hold more than one object with probability

$$1 - \frac{m!}{(m - n)!m^n}$$
Generalized Pigeonhole Principle: Example

• Among 10 people, what is the probability that two or more will have the same birthday?
  – Here n=10 and m=365 (ignoring leap years)
  – Thus, the probability that two will have the same birthday is

\[
1 - \frac{365!}{(365 - 10)!365^{10}} \approx 0.1169
\]

So, less than 12% probability
Pigeonhole Principle: Example A (1)

• Show that
  – in a room of n people with certain acquaintances,
  – some pair must have the same number of acquaintances

• Note that this is equivalent to showing that any symmetric, irreflexive relation on n elements must have two elements with the same number of relations

• Proof: by contradiction using the pigeonhole principle

• Assume, to the contrary, that every person has a different number of acquaintances: 0, 1, 2, ..., n-1 (no one can have n acquaintances because the relation is irreflexive).

• There are n possibilities, we have n people, we are not done 😞
Pigeonhole Principle: Example A (2)

• Assume, to the contrary, that every person has a different number of acquaintances: 0, 1, 2, ..., n-1 (no one can have n acquaintances because the relation is irreflexive).
• There are n possibilities, we have n people, we are not done 😞
• We need to use the fact that acquaintanceship is a symmetric irreflexive relation
• In particular, some person knows 0 people while another knows n-1 people
• This is impossible. Contradiction! 😊 So we do not have n (10) possibilities, but less
• Thus by the pigeonhole principle (10 people and 9 possibilities) at least two people have to the same number of acquaintances
Pigeonhole Principle: Example B

- **Example**: Say, 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats.
- **Show that**
  - One of the buses will have 14 empty seats
  - One of the buses will carry at least 67 passengers
- **One of the buses will have 14 empty seats**
  - Total number of seats is $80 \times 30 = 2400$
  - Total number of empty seats is $2400 - 2000 = 400$
  - By the pigeonhole principle: 400 empty seats in 30 buses, one must have $\lceil 400/30 \rceil = 14$ empty seats
- **One of the buses will carry at least 67 passengers**
  - By the pigeonhole principle: 2000 passengers in 30 buses, one must have $\lceil 2000/30 \rceil = 67$ passengers
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Permutations

• A permutation of a set of distinct objects is an ordered arrangement of these objects.

• An ordered arrangement of $r$ elements of a set of $n$ elements is called an $r$-permutation

• **Theorem:** The number of $r$ permutations of a set of $n$ distinct elements is

$$P(n, r) = \prod_{i=0}^{r-1} (n - i) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

• It follows that

$$P(n, r) = \frac{n!}{(n - r)!}$$

• In particular

$$P(n, n) = n!$$

• Note here that the order is important. It is necessary to distinguish when the order matters and it does not
Application of PIE and Permutations: Derangements (I) (Section 7.6)

• Consider the hat-check problem
  – Given
    • An employee checks hats from n customers
    • However, s/he forgets to tag them
    • When customers check out their hats, they are given one at random
  – Question
    • What is the probability that no one will get their hat back?
Application of PIE and Permutations: Derangements (II)

- The hat-check problem can be modeled using derangements: permutations of objects such that no element is in its original position.
  - Example: 21453 is a derangement of 12345 but 21543 is not.
- The number of derangements of a set with n elements is
  \[ D_n = n! \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!} \right] \]
- Thus, the answer to the hat-check problem is \( \frac{D_n}{n!} \).
- Note that \( e^{-1} = \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \ldots (-1)^n \frac{1}{n!} \right] \).
- Thus, the probability of the hat-check problem converges
  \[ \lim_{n \to \infty} \frac{D_n}{n!} = e^{-1} \approx 0.368 \]

See textbook, Section 8.6 page 562

CSCE 235 Combinatorics
Permutations: Example A

• How many pairs of dance partners can be selected from a group of 12 women and 20 men?
  – The first woman can partner with any of the 20 men, the second with any of the remaining 19, etc.
  – To partner all 12 women, we have
    \[ P(20,12) = \frac{20!}{8!} = 9 \times 10 \times 11 \ldots 20 \]
Permutations: Example B

- In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?
  - The number of ways is $P(24,10)$
  - Since we can choose either a or z to come first, then there are $2P(24,10)$ arrangements of the 12-letter block
  - For the remaining 14 letters, there are $P(15,15)=15!$ possible arrangements
  - In all there are $2P(24,10).15!$ arrangements
Permutations: Example C (1)

- How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern $bge$ nor $eaf$?
  
  - The total number of permutations is $P(7,7) = 7!$

  - If we fix the pattern $bge$, then we consider it as a single block. Thus, the number of permutations with this pattern is $P(5,5) = 5!$

  - If we fix the pattern $bge$, then we consider it as a single block. Thus, the number of permutations with this pattern is $P(5,5) = 5!$

  - Fixing the pattern $eaf$, we have the same number: 5!

  - Thus, we have $(7! – 2.5!)$. Is this correct?

  - No! we have subtracted too many permutations: ones containing both $eaf$ and $bfe$. 
Permutations: Example C (2)

- There are two cases: (1) eaf comes first, (2) bge comes first

- Are there any cases where eaf comes before bge?

- No! The letter e cannot be used twice

- If bge comes first, then the pattern must be bgeaf, so we have 3 blocks or 3! arrangements

- Altogether, we have

\[7! - 2 \cdot (5!) + 3! = 4806\]
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Combinations (1)

- Whereas permutations consider order, combinations are used when order does not matter.

- **Definition:** A k-combination of elements of a set is an unordered selection of k elements from the set.
  
  (A combination is imply a subset of cardinality k)
Combinations (2)

• **Theorem**: The number of $k$-combinations of a set of cardinality $n$ with $0 \leq k \leq n$ is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n - k)!k!}$$

is read ‘$n$ choose $k$’.  

\[\binom{n}{k}\]
Combinations (3)

- A useful fact about combinations is that they are symmetric
  \[
  \binom{n}{1} = \binom{n}{n-1} \quad \binom{n}{2} = \binom{n}{n-2} \quad \binom{n}{3} = \binom{n}{n-3}
  \]

- Corollary: Let \( n, k \) be nonnegative integers with \( k \leq n \), then
  \[
  \binom{n}{k} = \binom{n}{n-k}
  \]
Combinations: Example A

• In the Powerball lottery, you pick
  – five numbers between 1 and 55 and
  – A single ‘powerball’ number between 1 and 42
How many possible plays are there?

• Here order does not matter
  – The number of ways of choosing 5 numbers is \( \binom{55}{5} \)
  – There are 42 possible ways to choose the powerball
  – The two events are not mutually exclusive: \( 42 \binom{55}{5} \)
  – The odds of winning are \( \frac{1}{42 \binom{55}{5}} < 0.000000006845 \)
Combinations: Example B

- In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?
  - The number of ways of choosing 3 heads out of 10 coin tosses is \( \binom{10}{3} \)
  - It is the same as choosing 7 tails out of 10 coin tosses \( \binom{10}{7} = \binom{10}{3} = 120 \)
  - ... which illustrates the corollary \( \binom{n}{k} = \binom{n}{n-k} \)
Combinations: Example C

- How many committees of 5 people can be chosen from 20 men and 12 women
  - If exactly 3 men must be on each committee?
  - If at least 4 women must be on each committee?

- **If exactly three men must be on each committee?**
  - We must choose 3 men and 2 women. The choices are **not** mutually exclusive, we use the product rule
    \[
    \binom{20}{3} \cdot \binom{12}{2}
    \]

- **If at least 4 women must be on each committee?**
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. Theses choices are mutually exclusive, we use the addition rule:
    \[
    \binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692
    \]
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Binomial Coefficients (1)

- The number of $r$-combinations \( \binom{n}{r} \) is also called the 
  binomial coefficient

- The binomial coefficients are the coefficients in the 
  expansion of the expression, (multivariate polynomial), 
  \((x+y)^n\)

- A binomial is a sum of two terms
Theorem: Binomial Theorem

Let $x$, $y$, be variables and let $n$ be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Expanding the summation we have

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example

$$(x + y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3$$
Binomial Coefficients: Example

• What is the coefficient of the term $x^8y^{12}$ in the expansion of $(3x+4y)^{20}$?
  
  – By the binomial theorem, we have
    $$(3x + 4y)^{20} = \sum_{j=0}^{20} \binom{20}{j} (3x)^{20-j} (4y)^j$$

  – When $j=12$, we have
    $$\binom{20}{12} (3x)^8 (4y)^{12}$$

  – The coefficient is
    $$\binom{20}{12} 3^8 4^{12} = \frac{20!}{12!8!} 3^8 4^{12} = 13866187326750720$$
Binomial Coefficients (3)

- Many useful identities and facts come from the Binomial Theorem
- **Corollary:**
  \[
  \sum_{k=0}^{n} \binom{n}{k} = 2^n \\
  \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad n \geq 1 \\
  \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n 
  \]

*Equalities are based on \((1+1)^n=2^n\), \((-1+1)^n=0^n\), \((1+2)^n=3^n\)*
Binomial Coefficients (4)

• **Theorem:** Vandermonde’s Identity
  Let $m, n, r$ be nonnegative integers with $r$ not exceeding either $m$ or $n$. Then
  \[
  \binom{m + n}{r} = \sum_{k=0}^{r} \binom{m}{r - k} \binom{n}{k}
  \]

• **Corollary:** If $n$ is a nonnegative integer then
  \[
  \binom{2n}{n} = \sum_{k=0}^{r} \binom{n}{k}^2
  \]

• **Corollary:** Let $n, r$ be nonnegative integers, $r \leq n$, then
  \[
  \binom{n + 1}{r + 1} = \sum_{j=r}^{n} \binom{j}{r}
  \]
Binomial Coefficients: Pascal’s Identity & Triangle

- The following is known as Pascal’s identity which gives a useful identity for efficiently computing binomial coefficients

- **Theorem:** Pascal’s Identity
  Let \( n, k \in \mathbb{Z}^+ \) with \( n \geq k \), then
  \[
  \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
  \]

  Pascal’s Identity forms the basis of a geometric object known as Pascal’s Triangle
Pascal’s Triangle

\[
\begin{array}{ccc}
\binom{0}{0} & \binom{0}{1} & \binom{1}{1} \\
\binom{1}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{2}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\binom{3}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
\binom{4}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
\]
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  – Generating combinations (1), permutations (2)
• More Examples
Generalized Combinations & Permutations (1)

• Sometimes, we are interested in permutations and combinations in which repetitions are allowed

• **Theorem:** The number of r-permutations of a set of n objects with repetition allowed is $n^r$
  
  ...which is easily obtained by the product rule

• **Theorem:** There are

$$\binom{n + r - 1}{r}$$

r-combinations from a set with n elements when repetition of elements is allowed
Generalized Combinations & Permutations:  
Example

• There are 30 varieties of donuts from which we wish to buy a dozen. How many possible ways to place your order are there?

• Here, \( n=30 \) and we wish to choose \( r=12 \).

• Order does not matter and repetitions are possible

• We apply the previous theorem

• The number of possible orders is

\[
\binom{n + r - 1}{r} = \binom{30 + 12 - 1}{12} = \binom{17}{12}
\]
Generalized Combinations & Permutations (2)

• **Theorem:** The number of different permutations of n objects where there are $n_1$ indistinguishable objects of type 1, $n_2$ of type 2, and $n_k$ of type k is

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$

An equivalent ways of interpreting this theorem is the number of ways to

– distribute n distinguishable objects
– into k distinguishable boxes
– so that $n_i$ objects are place into box i for i=1,2,3,...,k
Example

• How many permutations of the word Mississippi are there?
• ‘Mississippi’ has
  – 4 distinct letters: m,i,s,p
  – with 1,4,4,2 occurrences respectively
  – Therefore, the number of permutations is

\[
\frac{11!}{1!4!4!2!}
\]
Outline

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• Counting:
  – Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  – Application of PIE: Number of onto functions
• Pigeonhole principle
  – Generalized, probabilistic forms
• Permutations
• Combinations
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  – *Generating combinations (1), permutations (2)*
• More Examples
Algorithms

• In general, it is inefficient to solve a problem by considering all permutation or combinations since there are exponential (worst, factorial!) numbers of such arrangements.

• Nevertheless, for many problems, no better approach is known.

• When exact solutions are needed, backtracking algorithms are used to exhaustively enumerate all arrangements.
Algorithms: Example

- **Traveling Salesperson Problem (TSP)**
  Consider a salesman that must visit \( n \) different cities. He wishes to visit them in an order such that his overall distance travelled is minimized.

- This problem is one of hundred of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that no efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP.)

- The only way of solving this problem exactly is to try all possible \( n! \) routes.

- We give several algorithms for generating these combinatorial objects.
Generating Combinations (1)

- Recall that combinations are simply all possible subsets of size \( r \). For our purposes, we will consider generating subsets of \( \{1,2,3,...,n\} \)

- The algorithm works as follows
  - Start with \( \{1,...,r\} \)
  - Assume that we have \( a_1a_2...a_r \), we want the next combination
  - Locate the last element \( a_i \) such that \( a_i \neq n-r-l \)
  - Replace \( a_i \) with \( a_i+1 \)
  - Replace \( a_j \) with \( a_i+j-l \) for \( j=i+1, i+2,...,r \)
Generating Combinations (2)

Next r-Combinations

Input: A set of n elements and an r-combination \( a_1, a_2, ..., a_r \)

Output: The next r-combination

1. \( i \leftarrow r \)
2. While \( a_i = n-r+i \) Do
3. \( i \leftarrow i-1 \)
4. End
5. \( a_i \leftarrow a_i + 1 \)
6. For \( j \leftarrow (i+1) \) to \( r \) Do
7. \( a_j \leftarrow a_i + j - i \)
8. End
Generating Combinations: Example

• Find the next 3-combination of the set \{1,2,3,4,5\} after \{1,4,5\}
• Here \(a_1=1\), \(a_2=4\), \(a_3=5\), \(n=5\), \(r=3\)
• The last \(i\) such that \(a_i \neq 5-3+i\) is 1

• Thus, we set
  \[a_1 = a_1 + 1 = 2\]
  \[a_2 = a_1 + 2 - 1 = 3\]
  \[a_3 = a_1 + 3 - 1 = 4\]

Thus, the next \(r\)-combinations is \{2,3,4\}
Generating Permutations

• The textbook gives an algorithm to generate permutations in lexicographic order. Essentially, the algorithm works as follows. Given a permutation
  – Choose the left-most pair $a_j, a_{j+1}$ where $a_j < a_{j+1}$
  – Choose the least items to the right of $a_j$ greater than $a_j$
  – Swap this item and $a_j$
  – Arrange the remaining (to the right) items in order
**Next Permutation** (lexicographic order)

**INPUT**
A set of \( n \) elements and an \( r \)-permutation, \( a_1 \cdots a_r \).

**OUTPUT**
The next \( r \)-permutation.

1. \( j = n - 1 \)
2. **WHILE** \( a_j > a_{j+1} \) **DO**
3. \( j = j - 1 \)
4. **END**

   // \( j \) is the largest subscript with \( a_j < a_{j+1} \)

5. \( k = n \)
6. **WHILE** \( a_j > a_k \) **DO**
7. \( k = k - 1 \)
8. **END**

   // \( a_k \) is the smallest integer greater than \( a_j \) to the right of \( a_j \)

9. \( \text{swap}(a_j, a_k) \)
10. \( r = n \)
11. \( s = j + 1 \)
12. **WHILE** \( r > s \) **DO**
13. \( \text{swap}(a_r, a_s) \)
14. \( r = r - 1 \)
15. \( s = s + 1 \)
16. **END**
Generating Permutations (2)

• Often there is no reason to generate permutations in lexicographic order. Moreover even though generating permutations is inefficient in itself, lexicographic order induces even more work.

• An alternate method is to fix an element, then recursively permute the n-1 remaining elements.

• The Johnson-Trotter algorithm has the following attractive properties. Not in your textbook, not on the exam, just for your reference/culture:
  – It is bottom up (non-recursive)
  – It induces a minimal-change between each permutation
Johnson-Trotter Algorithm

• We associate a direction to each element, for example

\begin{align*}
\begin{array}{cccc}
3 & 2 & 4 & 1 \\
\end{array}
\end{align*}

• A component is mobile if its direction points to an adjacent component that is smaller than itself.

• Here 3 and 4 are mobile, 1 and 2 are not
Algorithm: Johnson Trotter

\textbf{Input} : An integer \( n \).

\textbf{Output} : All possible permutations of \( \langle 1, 2, \ldots, n \rangle \).

1 \quad \pi = \frac{1}{1} \frac{2}{2} \ldots \frac{n}{n}

2 \quad \textbf{while} \ There \ exists \ a \ mobile \ integer \ \( k \in \pi \) \ \textbf{do}

3 \quad k = \text{largest mobile integer}

4 \quad \text{swap} \ k \ \text{and the adjacent integer} \ k \ \text{points to}

5 \quad \text{reverse direction of all integers} \ > \ k

6 \quad \text{Output} \ \pi

7 \quad \text{END}
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Example A

• How many bit strings of length 4 are there such that 11 never appear as a substring
• We can represent the set of strings graphically using a diagram tree (see textbook pages 395)
Example: Counting Functions (1)

- Let \( S, T \) be sets such that \( |S| = n, \quad |T| = m \).
  - How many function are there mapping \( f:S \rightarrow T \)?
  - How many of these functions are one-to-one (injective)?
- A function simply maps each \( s_i \) to one \( t_j \), thus for each \( n \) we can choose to send it to any of the elements in \( T \)
- Each of these is an independent event, so we apply the multiplication rule:
- If we wish \( f \) to be injective, we must have \( n \leq m \), otherwise the answer is obviously 0
Example: Counting Functions (2)

- Now each $s_i$ must be mapped to a unique element in $T$.
  - For $s_1$, we have $m$ choices.
  - However, once we have made a mapping, say $s_j$, we cannot map subsequent elements to $t_j$ again.
  - In particular, for the second element, $s_2$, we now have $m-1$ choices, for $s_3$, $m-2$ choices, etc.

$$m \cdot (m - 1) \cdot (m - 2) \cdot \ldots (m - (n - 2)) \cdot (m - (n - 1))$$

- An alternative way of thinking is using the choose operator: we need to choose $n$ elements from a set of size $m$ for our mapping

$$\binom{m}{n} = \frac{m!}{(m - n)!n!}$$

- Once we have chosen this set, we now consider all permutations of the mapping, that is $n!$ different mappings for this set. Thus, the number of such mapping is

$$\frac{m!}{(m - n)!n!} \cdot n! = \frac{m!}{(m - n)!}$$

Combinatorics
Another Example: Counting Functions

• Let $S=\{1,2,3\}$, $T=\{a,b\}$.
  – How many onto (surjective) mappings are there from $S\rightarrow T$?
  – How many onto-to-one injective functions are there from $T\rightarrow S$?
• See Theorem 1, page 561
Example: Sets

• How many k integers $1 \leq k \leq 100$ are divisible by 2 or 3?

• Let
  
  - $A = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (2 \mid n)\}$
  - $B = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (3 \mid n)\}$

• Clearly, $|A| = \left\lceil \frac{100}{2} \right\rceil = 50$, $|B| = \left\lceil \frac{100}{3} \right\rceil = 33$

• Do we have $|A \cup B| = 83$? No!

• We have over counted the integers divisible by 6
  
  - Let $C = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \land (6 \mid n)\}$, $|C| = \left\lceil \frac{100}{6} \right\rceil = 16$

• So $|A \cup B| = (50+33) - 16 = 67$
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