Week 9 Recitation

Robert Woodward

March 7, 2011

- Questions about lecture / homework so far?
- Rosen 8.4.25(b). Use Algorithm 1 (Given on page 550) to compute the transitive closure of the relation $\{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$ on the set $\{1,2,3,4\}$.
  - We note the matrix of a relation R^x resulting from the composing the relation R with itself x times: M_{R^x}, alternatively: $M_R^{[x]}$.
  - We note the relations composition operator $\circ$ and the matrix product operator $\cdot$, alternatively, $\odot$.

$$M_R = M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = M_{R^1 \circ R^1} = M_{R^1} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R^3} = M_{R \circ R^2} = M_{R^2} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M_{R^4} = M_{R \circ R^3} = M_{R^3} \cdot M_{R^1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R^*} = M_{R^1} \vee M_{R^2} \vee M_{R^3} \vee M_{R^4}$$

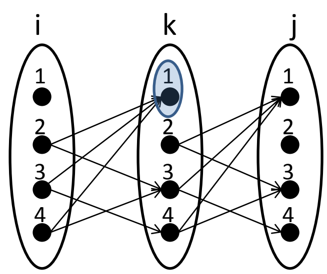
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

- (On request): Rosen 8.4.27(b). Use Warshall's algorithm to compute the transitive closure of the relation $\{(2,1),(2,3),(3,1),(3,4),(4,1),(4,3)\}$ on the set $\{1,2,3,4\}$.

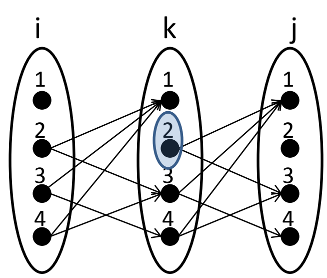
  $$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

  Fix $k = 1$:



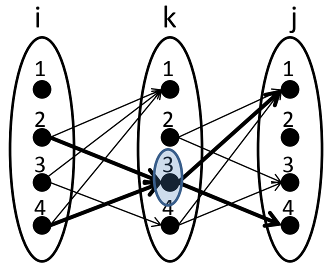
$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

  Fix $k = 2$:



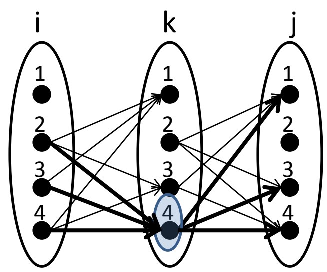
$$W_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Fix $k = 3$:



$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \mathbf{1} \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & \mathbf{1} \end{bmatrix}$$

Fix $k = 4$:



$$W_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & \mathbf{1} & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

- Rosen 8.5.3

  To be an equivalence relation, a relation must be:

  - Reflexive
  - Symmetric
  - Transitive

  Consider a function $f{:}\mathbb{Z} \longrightarrow \mathbb{Z}$.
  Now, consider the set F of all such functions, which becomes a quite abstract notion.

$$F = \{f \mid f : \mathbb{Z} \longrightarrow \mathbb{Z}\}.$$

  We are going to define a relation on this set F, let call it R

$$R \subseteq F \times F.$$

  Two functions $f, g \in F$ are related by R iff $(f, g) \in R$. We write fRg.

  The question is to determine whether or not each of the following relations on the set F is an equivalence relation.

a) Consider the relation: $\{(f,g)|f(1) = g(1)\}$. This means (f,g) is an element of R if and only if $f(1)$ and $g(1)$ evaluate to the same value in $\mathcal{Z}$.

This relation is reflexive because $\forall f \in F, f(1) = f(1)$.

This relation is symmetric because $\forall f, g \in F, fRg \rightarrow f(1) = g(1) \rightarrow g(1) = f(1) \rightarrow gRf$.

This relation is transitive because fRg and gRh, $\rightarrow (f(1) = g(1)) \wedge (g(1) = h(1)) \rightarrow f(1) = h(1) \rightarrow fRh$.

Therefore, R is an equivalence relation.

b) Consider the relation: $\{(f,g)|(f(0) = g(0)) \vee (f(1) = g(1))\}$.

This relation is reflexive and symmetric (by similar argument to part a), but it is *not* transitive.

Indeed, if

$$(fRg) \wedge (gRh) \Rightarrow (f(0) = g(0) \vee f(1) = g(1)) \wedge (g(0) = h(0) \vee g(1) = h(1)).$$

Consider the counterexample where

$$(f(0) = g(0)) \wedge (f(1) \neq g(1))$$

and

$$(g(0) \neq h(0)) \wedge (g(1) = h(1)).$$

We notice that $(f(0) = g(0) \neq h(0)) \wedge (f(1) \neq g(1) = h(1)) \Rightarrow (f(0) \neq h(0)) \wedge (f(1) \neq h(1)) \Rightarrow (f,h) \notin R$, and thus the relation is not transitive.

- Rosen 8.5.41(a): Is this collection of subsets of $\{1,2,3,4,5,6\}$ a partition? $\{1,2\}, \{2,3,4\}, \{4,5,6\}$.

  The non-empty set subsets $A_1, A_2, \ldots, A_k$ are a partition of a set A if and only if:

  - $\cup_{i=1}^{k} A_i = A$
  - $\forall i \neq j, A_i \cap A_j = \emptyset$
  - $\forall i, A_i \neq \emptyset$

  For our partition:

  - $\cup_{i=1}^{k} A_i = \{1,2\} \cup \{2,3,4\} \cup \{4,5,6\} = \{1,2,3,4,5,6\} = A$
  - $(\{1,2\} \cap \{2,3,4\} = \{2\} \neq \emptyset)$, $(\{2,3,4\} \cap \{4,5,6\} = \{4\} \neq \emptyset)$, $(\{1,2\} \cap \{4,5,6\} = \emptyset)$.
  - $(\{1,2\} \neq \emptyset)$, $(\{2,3,4\} \neq \emptyset)$ $(\{4,5,6\} \neq \emptyset)$.

  Because $\forall i \neq j, A_i \cap A_j \neq \emptyset$, the two sets do not form a partition of A.

- Rosen 8.5.41(b): Is this collection of subsets of $\{1,2,3,4,5,6\}$ a partition? $\{1\},\{2,3,6\},\{4\},\{5\}$.

  For our partition:

  - $\cup_{i=1}^{k} A_i = \{1\} \cup \{2,3,6\} \cup \{4\} \cup \{5\} = \{1,2,3,4,5,6\} = A$
  - $(\{1\} \cap \{2,3,6\} = \emptyset)$, $(\{1\} \cap \{4\} = \emptyset)$, $(\{1\} \cap \{5\} = \emptyset)$, $(\{2,3,6\} \cap \{4\} = \emptyset)$, $(\{2,3,6\} \cap \{5\} = \emptyset)$, $(\cap\{4\} \cap \{5\} = \emptyset)$.
  - $(\{1\} \neq \emptyset)$, $(\{2,3,6\} \neq \emptyset)$, $(\{4\} \neq \emptyset)$, $(\{5\} \neq \emptyset)$,

  Therefore, the two sets form a partition of A.

- Quiz (Last 10 minutes)

Extra material:

- Rosen 8.5.1(a)

  Our relation is: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  Our relation is reflexive, symmetric, and transitive. Therefore, it is an equivalence relation.

- Rosen 8.5.1(b)

  Our relation is: $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

  It is *not* reflexive (why?) and is *not* transitive (why?). Therefore, it is *not* an equivalence relation.

  Notice that for transitivity, in this case, it helps to draw the directed graph of the relation.