Questions about lecture / homework so far?

Rosen 2.3:19, but determine whether \( f \) is injective, surjective, bijective, and invertible. If \( f \) is not invertible, what is the largest domain and co-domain in which \( f \) is invertible.

\( f \) is defined over \( \mathbb{R} \to \mathbb{R} \).

a) \( f(x) = 2x + 1 \)

1. We prove that \( f \) is injective by applying the definition: \( f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2 \), therefore it is injective.
2. Surjective: \( \forall b \in \text{co-domain}(f), b = 2a + 1 \Rightarrow 2a = b - 1 \Rightarrow a = \frac{b-1}{2} \).

\( b \in \mathbb{R} \Rightarrow \frac{b-1}{2} \in \mathbb{R} \Rightarrow a \in \mathbb{R} = \text{domain}(f) \).

Thus, \( \forall b \in \text{co-domain}(f) \Rightarrow (\exists a \in \text{domain}(f), b = f(a)) \) Therefore, \( f \) is surjective.

3. Bijective: \( f \) is both injective and surjective, thus it is surjective.

4. \( f \) is bijective thus it is invertible: \( f^{-1}(y) = x \). We know that \( y = 2x + 1 \Rightarrow x = \frac{y-1}{2} \). Thus, \( f^{-1}(y) = \frac{y-1}{2} \).

b) \( f(x) = x^2 + 1 \)

1. Surjective: For some element \( b \in \text{co-domain}(f), b = a^2 + 1 \Rightarrow \pm \sqrt{b-1} = a \), for some element \( a \in \mathbb{R} = \text{domain}(f) \). But, if \( b < 1 \), \( \sqrt{b-1} \) is not defined. Therefore, \( f \) is not surjective.

Could have also proved that \( f \) is not surjective by using a proof by counterexample: \(-4 = f(x) \) has no such pre-image.

If we were to restrict the co-domain to the set of all reals greater than or equal to 1, \( \sqrt{b-1} \) will be defined. Thus, \( \forall b \in \mathbb{R} \geq 1 \Rightarrow (\exists a \in \text{domain}(f), b = f(a)) \). Therefore, on the restricted co-domain, \( f \) is surjective.

2. We try to prove that \( f \) is injective by applying the definition: \( f(x_1) = f(x_2) \Rightarrow x_1^2 + 1 = x_2^2 + 1 \). Now remember the trick mentioned repeatedly in class: we move the same ‘powers’ to the same side: \( f(x_1) = f(x_2) \Rightarrow x_1^2 - x_2^2 = 1 - 1 = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0 \). Now assume \( x_1 \neq x_2 \) and dividing by \( (x_1 - x_2) \), we get: \( f(x_1) = f(x_2) \Rightarrow x_1 + x_2 = 0 \) which is possible
for an infinity of pair of values $x_1, x_2 \in \text{domain}(f)$ with $x_1 = -x_2$ such as 2,-2 or 10,-10. Therefore, $f$ is not injective.

Could have also proved that $f$ is not injective by using a proof by counter-example: $4 = f(x)$ has two pre-images $x = -2$ and $x = 2$.

If we were to restrict the domain to the set of all reals greater or equal to 1, the function can only be satisfied when $x_1 = x_2$. Therefore, on the restricted domain, $f$ is injective.

3. Bijective: No, because $f$ is not injective. On the restricted co-domain and domain, $f$ will be bijective.

4. For the restricted domain, $f$ is bijective thus it is invertible: $f^{-1}(y) = x$.

We know that $y = x^2 + 1 \Rightarrow x = \sqrt{y-1}$. Thus, $f^{-1}(y) = \sqrt{y-1}$. (Note that really $x = \pm \sqrt{y-1}$. Though, because the restricted domain, $x$ must be positive).

Let $f(x) = x - 4$ and $g(x) = (x + 1)^2 + 1$. What is $f \circ g$?

Formally, you need first to make sure that

1. $f, g$ are functions.
2. $\text{rng}(g) \subseteq \text{domain}(f)$.

Then you can proceed with the calculations: $f \circ g = f(g(x)) = f((x + 1)^2 + 1) = (x + 1)^2 + 1 - 4 = x^2 + 2x + 1 + 1 - 4 = x^2 + 2x - 4$.

• (Parts from Rosen 8.1:3) Let:

- $S = \{1, 2, 3, 4\}$, and let $R_1$ and $R_2$ be defined on $S$.
- $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $R_2 = \{(1, 2), (2, 3), (3, 4)\}$

What is:

- The 0-1 matrix representation of $R_1$?

$$M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

- Is $R_1$ irreflexive? No, it is reflexive because $\forall i M_{i,i} = 1$
- Is $R_1$ transitive? Yes, because $\forall i, j, k (M_{i,j} = 1) \land (M_{j,k} = 1) \land (M_{i,k} = 1)$.
- Is $R_1$ asymmetric? No, because $M_{1,1} = M_{1,1}$
- Is $R_1$ antisymmetric? Yes, because $\forall i, j (M_{i,j} \land M_{j,i}) \rightarrow (i = j)$
The 0-1 matrix representation of $R_2$?
\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The 0-1 matrix representation of $R_1 \cup R_2$?
\[
N = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Is $R_1 \cup R_2$ irreflexive? No, it is reflexive because $\forall i \, N_{i,i} = 1$

Is $R_1 \cup R_2$ transitive? No, because $N_{1,2} = 1 \wedge N_{2,3} = 1$, but $N_{1,3} = 0$.

Is $R_1 \cup R_2$ antisymmetric? Yes, because $\forall i, j \, (N_{i,j} \wedge N_{j,i}) \rightarrow (i = j)$.

Is $R_1 \cup R_2$ asymmetric? No, because $N_{1,1} = N_{1,1}$.

Is $R_1 \cup R_2$ symmetric? No, because $N_{1,2} \neq N_{2,1}$.

The 0-1 matrix representation of $R_1 \circ R_2$? (Note: You apply $R_2$ then $R_1$.)

Using the graphical representation:

\[
M_{R_1 \circ R_2} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

You can also compute $M_{R_1 \circ R_2}$ by doing the bit matrix product of $M_{R_2} \cdot M_{R_1}$.

No Quiz...