Week 8 Recitation

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- Questions about lecture / homework so far?
- Rosen 2.3:19, but determine whether f is injective, surjective, bijective, and invertible. If f is not invertible, what is the largest domain and co-domain in which f is invertible).

  f is defined over $\mathcal{R} \to \mathcal{R}$.

  a) $f(x) = 2x + 1$

  1. We prove that f is injective by applying the definition: $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$, therefore it is injective.
  2. Surjective: $\forall b \in \text{co} - \text{domain}(f)$, $b = 2a + 1 \Rightarrow 2a = b - 1 \Rightarrow a = \frac{b-1}{2}$.
     $b \in \mathcal{R} \Rightarrow \frac{b-1}{2} \in \mathcal{R} \Rightarrow a \in \mathcal{R} = \text{domain}(f)$.
     Thus, $\forall b \in \text{co} - \text{domain}(f) \Rightarrow (\exists a \in \text{domain}(f), b = f(a))$ Therefore, f is surjective.
  3. Bijective: f is both injective and surjective, thus it is surjective.
  4. f is bijective thus it is invertible: $f^{-1}(y) = x$. We know that $y = 2x + 1 \Rightarrow x = \frac{y-1}{2}$. Thus, $f^{-1}(y) = \frac{y-1}{2}$.

  b) $f(x) = x^2 + 1$

  1. Surjective: For some element $b \in \text{co} - \text{domain}(f)$, $b = a^2 + 1 \Rightarrow \pm\sqrt{b-1} = a$, for some element $a \in \mathcal{R} = \text{domain}(f)$. But, if $b < 1$, $\sqrt{b-1}$ is not defined. Therefore, f is not surjective.
     Could have also proved that f is not surjective by using a proof by counter-example: $-4 = f(x)$ has no such pre-image.
     If we were to restrict the co-domain to the set of all reals greater than or equal to 1, $\sqrt{b-1}$ will be defined. Thus, $\forall b \in \mathcal{R} \geq 1 \Rightarrow (\exists a \in \text{domain}(f), b = f(a))$. Therefore, on the restricted co-domain, f is surjective.
  2. We try to prove that f is injective by applying the definition: $f(x_1) = f(x_2) \Rightarrow x_1^2 + 1 = x_2^2 + 1$. Now remember the trick mentioned repeatedly in class: we move the same 'powers' to the same side: $f(x_1) = f(x_2) \Rightarrow x_1^2 - x_2^2 = 1 - 1 = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$. Now assume $x_1 \neq x_2$ and dividing by $(x_1 - x_2)$, we get: $f(x_1) = f(x_2) \Rightarrow x_1 + x_2 = 0$ which is possible

for an infinity of pair of values $x_1, x_2 \in$ domain(f) with $x_1 = -x_2$ such as 2,-2 or 10,-10. Therefore, f is not injective.
Could have also proved that f is not injective by using a proof by counter-example: $4 = f(x)$ has two pre-images $x = -2$ and $x = 2$.
If we were to restrict the domain to the set of all reals greater or equal to 1, the function can only be satisfied when $x_1 = x_2$. Therefore, on the restricted domain, f is injective.

3. Bijective: No, because f is not injective.
   On the restricted co-domain and domain, f will be bijective.
4. For the restricted domain, f is bijective thus it is invertible: $f^{-1}(y) = x$. We know that $y = x^2 + 1 \Rightarrow x = \sqrt{y-1}$. Thus, $f^{-1}(y) = \sqrt{y-1}$. (Note that really $x = \pm\sqrt{y-1}$. Though, because the restricted domain, x must be positive).

- Let $f(x) = x - 4$ and $g(x) = (x+1)^2 + 1$. What is $f \circ g$?

  Formally, you need first to make sure that

  1. f, g are functions.
  2. rng(g) $\subseteq$ domain(f).

  Then you can proceed with the calculations: $f \circ g = f(g(x)) = f((x+1)^2 + 1) = (x+1)^2 + 1 - 4 = x^2 + 2x + 1 + 1 - 4 = x^2 + 2x - 4$.

- (Parts from Rosen 8.1:3) Let:

  - $S = \{1, 2, 3, 4\}$, and let R_1 and R_2 be defined on S.
  - $R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$
  - $R_2 = \{(1,2), (2,3), (3,4)\}$

  What is:

  - The 0-1 matrix representation of R_1?
    $$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
  - Is R_1 irreflexive? No, it is reflexive because $\forall i M_{i,i} = 1$
  - Is R_1 transitive? Yes, because $\forall i, j, k (M_{i,j} = 1) \wedge (M_{j,k} = 1) \wedge (M_{i,k} = 1)$.
  - Is R_1 asymmetric? No, because $M_{1,1} = M_{1,1}$
  - Is R_1 antisymmetric? Yes, because $\forall i, j (M_{i,j} \wedge M_{j,i}) \rightarrow (i = j)$

– The 0-1 matrix representation of R_2?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

– The 0-1 matrix representation of $R_1 \cup R_2$?

$$N = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

– Is $R_1 \cup R_2$ irreflexive? No, it is reflexive because $\forall i N_{i,i} = 1$

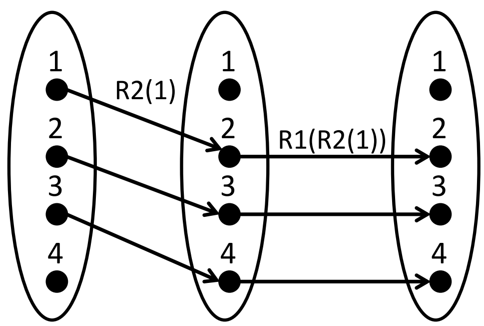
– Is $R_1 \cup R_2$ transitive? No, because $N_{1,2} = 1 \wedge N_{2,3} = 1$, but $N_{1,3} = 0$.

– Is $R_1 \cup R_2$ antisymmetric? Yes, because $\forall i, j(N_{i,j} \wedge N_{j,i}) \rightarrow (i = j)$.

– Is $R_1 \cup R_2$ asymmetric? No, because $N_{1,1} = N_{1,1}$.

– Is $R_1 \cup R_2$ symmetric? No, because $N_{1,2} \neq N_{2,1}$.

– The 0-1 matrix representation of $R_1 \circ R_2$? (Note: You apply R_2 then R_1.)
Using the graphical representation:



$$M_{R_1 \circ R_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

You can also compute $M_{R_1 \circ R_2}$ by doing the bit matrix product of $M_{R_2} \cdot M_{R_1}$.

$$M_{R_1 \circ R_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- No Quiz...