Week 14 Recitation

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- Questions about lecture / homework so far?
- Rosen 3.1.9, page 177. A palindrome is a string that reads the same forward and backwards. Describe an algorithm for determining whether a string of n characters is a palindrome.

Algorithm 1: PALINDROME: Checks if a string is a palindrome

Input: $a_1 a_2 a_3 \ldots a_n$, a string of length n
Output: If the string is a palindrome

$answer \leftarrow true$;
for $i \leftarrow 1$ ***to*** $\lfloor n/2 \rfloor$ **do**
  if $a_i \neq a_{n+1-i}$ **then**
    $answer \leftarrow false$;
  end
end
return $answer$;

Use the Mathematical Analysis of Algorithms to analyze PALINDROME's performance.

Recall the general strategy for the Mathematical Analysis of Algorithms:

1. Decide on a parameter(s) for the input, n
2. Identify the basic operation
3. Evaluate if the elementary operation depends only on n
4. Set up a summation corresponding to the number of elementary operations
5. Simplify the equation to get as simple of a function $f(n)$ as possible.

For PALINDROME:

1. The input parameter is n, the size of the string
2. The basic operation is the comparison of a_i and a_{n+1-i}

3. The elementary operation depends only on n
4. The summation for the number of elementary operations is: $\sum_{i=1}^{\lfloor n/2 \rfloor} 1$
5. First, recall that: $\sum_{i=m}^{n} 1 = n - m + 1$
   Therefore, $\sum_{i=1}^{\lfloor n/2 \rfloor} 1 = \lfloor n/2 \rfloor - 1 + 1 = \lfloor n/2 \rfloor$

- Reminder. To prove that $f(n) \in \Delta(g(n))$, we saw three main techniques:
  1. Applying the definition: Specify n_0 and c (or c_1 and c_2).
  2. The limit method: $lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f(n) \in \mathcal{O}(g(n)) \\ c > 0 & f(n) \in \Theta(g(n)) \\ \infty & f(n) \in \Omega(g(n)) \end{cases}$
  3. Applying the rule of L'Hôpital in the limit method: $lim_{n\to\infty} \frac{f(n)}{g(n)} = lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

- Let $f(n) = 3n^4 + 1$ and $g(n) = n^5 - 100$. Find a tight bound of the form $f(n) \in \Delta(g(n))$ and prove this bound formally.

  Intuitively we think that $f(n) \in \mathcal{O}(g(n))$ because the highest degree of $f(n)$ is lower than the highest degree of $g(n)$. Therefore, we want to show $f(n) \in \mathcal{O}(g(n))$.

  We need to find:

  – $c \in \mathbb{R}^+$
  – $n_0 \in \mathbb{N}$

  Such that for every positive integer $n \geq n_0$ we have $f(n) \leq cg(n)$

  (Note that for $n \geq 4$, $g(n) \geq 0$ because $4^5 - 100 = 1024 - 100 = 924$.)

  $\forall n \geq n_0 = 4$ we have:

$$3n^4 \leq 3n^5 \leq 3n^5 + n^5 - 400 = 4n^5 - 400 = 4(n^5 - 100)$$

  Also,

$$1 \leq n^5 - 100$$

  Adding up the above two expressions, we get

$$f(n) = 3n^4 + 1 \leq 4(n^5 - 100) + (n^5 - 100) = 5(n^5 - 100) = 5g(n)$$

  Therefore, $f(n) \leq 5g(n)$ $\forall n \geq n_0 = 4$ and $c = 5$. Consequently, $f(n) \in \mathcal{O}(g(n))$.

- (Similar to Rosen 3.2, problem 11) Let $f(n) = 3n^4 + 1$ and $g(n) = \frac{n^4}{2}$. Find a tight bound of the form $f(n) \in \Delta(g(n))$ and prove this bound formally.

  Intuitively we think that $f(n) \in \Theta(g(n))$ because both $f(n)$ and $g(n)$ have a n^4 term as their highest degree. Therefore, we need to show:

1. $f(n) \in \mathcal{O}(g(n))$ and
2. $f(n) \in \Omega(g(n))$.

1. *To show* $f(n) \in \mathcal{O}(g(n))$, we need to find:
   - $c_1 \in \mathbb{R}^+$
   - $n_0 \in \mathbb{N}$

   Such that for every positive integer $n \geq n_0$ we have $f(n) \leq c_1 g(n)$.
   $\forall n \geq n_0 = 1$ we have:

   $$3n^4 = 6\frac{n^4}{2}$$

   Also,

   $$1 \leq n^4 = 2\frac{n^4}{2}$$

   Adding up the above two expressions, we get:

   $$f(n) = 3n^4 + 1 \leq 6\frac{n^4}{2} + 2\frac{n^4}{2} = 8\frac{n^4}{2} = 8g(n)$$

   Therefore, $f(n) \leq 8g(n)$ $\forall n \geq n_0 = 1$ and $c_1 = 8$. Consequently, $f(n) \in \mathcal{O}(g(n))$.
2. *To show* $f(n) \in \Omega(g(n))$, we need to find:
   - $c_2 \in \mathbb{R}^+$
   - $n_0 \in \mathbb{N}$

   such that for every integer $n \geq n_0$ we have $f(n) \geq c_2 g(n)$.
   For $n_0 \geq 1$, we have:

   $$3n^4 \geq n^4 \geq \frac{n^4}{2}$$

   Obviously, we have:

   $$1 \geq 0$$

   Adding up the above two expressions, we get:

   $$f(n) = 3n^4 + 1 \geq \frac{n^4}{2} + 0 = \frac{n^4}{2} = g(n)$$

   Therefore, $f(n) \geq 1g(n)$ $\forall n \geq n_0 = 1$ and $c = 1$. Consequently, $f(n) \in \Omega(g(n))$.

Because $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$, then $f(n) \in \Theta(g(n))$.

- Similar to above, consider instead where $f(n) = 3n^4 - 1$ and $g(n) = \frac{n^4}{2}$. Find a tight bound of the form $f(n) \in \Delta(g(n))$ and prove this bound formally.

  1. *To show* $f(n) \in \mathcal{O}(g(n))$, it will follow very similar to above.
  2. *To show* $f(n) \in \Omega(g(n))$, we need to find:
     - $c_2 \in \mathbb{R}^+$
     - $n_0 \in \mathbb{N}$

     such that for every integer $n \geq n_0$ we have $f(n) \geq c_2 g(n)$.
     For $n_0 \geq 1$, we have:

     $$3n^4 - 1 = 6\frac{n^4}{2} - 1 = \frac{n^4}{2} + 5\frac{n^4}{2} - 1 \geq \frac{n^4}{2}$$

     (Note that, $5\frac{n^4}{2} - 1 \geq 0$ because $n_0 \geq 1$.)
     Therefore, we get:

     $$f(n) = 3n^4 - 1 \geq \frac{n^4}{2} = g(n)$$

     Therefore, $f(n) \geq 1g(n)\ \forall n \geq n_0 = 1$ and $c = 1$. Consequently, $f(n) \in \Omega(g(n))$.

  Because $f(n) \in \mathcal{O}(g(n))$ and $f(n) \in \Omega(g(n))$, then $f(n) \in \Theta(g(n))$.

- (Last 10 minutes) Quiz