A knight on a chessboard can move

– One space horizontally (in either direction) and two spaces vertically (in either direction) or
– Two spaces horizontally (in either direction) and one space vertically (in either direction).

Suppose that we have an infinite chessboard, made up of all squares \((i, j)\) where \(i\) and \(j\) are nonnegative integers. Use mathematical induction to show that a knight starting at \((0, 0)\) can visit every square using a finite sequence of moves. [Hint: Use induction on the variable \(n = i + j\).]

We want to prove that a knight can visit every square on the chessboard, using induction on the city block distance of the knight to a square. We complete the following four steps:

1. **State the propositional predicate:**
   
   \(P(n):\) A knight can reach all squares \((i, j)\) where \(i + j \leq n\).

   Notice here that the propositional predicate is defined for \(n\) and is not prefixe by the quantifier \(\forall n\).

2. **Basis Step:** We must form and verify the basis case.

   First, we propose the following basis case:
   
   \(P(2): \forall i, j \in \mathbb{N} \text{ and } (i + j \leq 2), \text{ a knight can reach square } (i, j).\)

   Notice that the basis case corresponds to a knight being able to reach several squares, all those where \((i + j \leq 2)\).

   Second, we need to verify that \(P(2)\) holds. We have 6 squares with \(i + j \leq 2\):
(a) (0, 0): This state is reachable because this is where we start.

(b) (0, 1): The knight moves from (0, 0) to (1, 2) to (2, 0) to (0, 1).

(c) (1, 0): The knight moves from (0, 0) to (2, 1) to (0, 2) to (1, 0).

(d) (2, 0): The knight reached this square on its way to (0, 1).

(e) (0, 2): The knight reached this square on its way to (1, 0).

(f) (1, 1): The knight moves from (0, 0) to (1, 2) to (0, 1) to (2, 2) to (0, 3) to (1, 1).

3. **State the Inductive Hypothesis:** We assume that $P(k)$ holds for all $k \geq 2$:

$$P(k): \text{A knight can reach all squares } (i, j) \text{ where } i + j \leq k.$$ 

4. **Inductive Step:** We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true. That is, we prove that $P(k+1)$ must hold assuming that $P(k)$ holds. Consider the statement $P(k)$:

$$P(k): \text{A knight can reach all squares } (i, j) \text{ where } i + j \leq k.$$ 

We want to show that a knight can reach all squares $(i', j')$ where $i' + j' \leq k + 1$. Because $k + 1 \geq 3$, at least one of $i'$ and $j'$ can be chosen at least 2. WLOG, assume $i' \geq 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i' - 2, j' + 1)$, because $i' - 2 + j' + 1 = i' + j' - 1 = k$. From such a square, it is just one move to the square $(i', j')$. Therefore, $P(k+1)$ holds, so we have shown that $P(k) \rightarrow P(k+1)$.

In conclusion, the statement holds by the Principle of Mathematical Induction. 

- Rosen 4.1, Problem 5, Page 280. (Similar to homework problem Rosen 4.1.4)

Using the Principle of Mathematical Induction, we will prove that

$$\forall n \in \mathbb{N}, 1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3.$$ 

We complete the following four steps:

1. **State the propositional predicate:**

$$P(n): 1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3.$$ 

2
2. **Basis Step:** We verify that $P(1)$ is true.
   
   We must form the basis step. First, we can write out what $P(1)$ is.
   
   $$P(1) : 1^2 + 3^2 = (1 + 1)(2 \times 1 + 1)(2 \times 1 + 3)/3.$$ 

   Second, we need to verify that this holds: 
   $$1 + 9 = (2)(2 + 1)(2 + 3)/3 \Rightarrow 10 = 2 \times 3 \times 5 \Rightarrow 10 = 10.$$ 

3. **State the Inductive Hypothesis:** We assume that $P(k)$ holds for all $k \geq 1$:
   
   $$P(k) : 1^2 + 3^2 + 5^2 + \ldots + (2k + 1)^2 = (k + 1)(2k + 1)(2k + 3)/3.$$ 

4. **Inductive Step:** We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers $k$. That is, we prove that $P(k + 1)$ must hold assuming that $P(k)$ holds for all positive integers $k$. Consider the statement $P(k)$:
   
   $$1^2 + 3^2 + \ldots + (2k + 1)^2 = (k + 1)(2k + 1)(2k + 3)/3.$$ 

   Adding the term $(2k + 3)^2$ on both sides of the equality, we get:
   
   $$1^2 + 3^2 + \ldots + (2k + 1)^2 + (2k + 3)^2 = (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2 = (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3) = (2k + 3)[(k + 1)(2k + 1)/3 + (2k + 3)] = (2k + 3)[(2k^2 + 3k + 1)/3 + (6k + 9)/3] = (2k + 3)[(2k^2 + 9k + 10)/3] = (2k + 3)(2k + 5)(k + 2)/3 = ([k + 1] + 1)(2[k + 1] + 1)(2[k + 1] + 3)/3.$$

   Therefore, $P(k+1)$ holds, so we have shown that $P(k) \rightarrow P(k+1)$. In conclusion, the statement holds by the Principle of Mathematical Induction. $\square$

- Rosen 4.1, Problem 33, page 280. (Similar to homework problem Rosen 4.1.32)

Prove that $5$ divides $n^5 - n$ whenever $n$ is a nonnegative integer.

We complete the following four steps:

1. **State the propositional predicate:**
   
   $$P(n) : 5|(n^5 - n).$$

2. **Basis Step:** We verify that $P(1)$ is true.

   We must form the basis step. First, we can write out what $P(0)$ is.
   
   $$P(0) : 5|(0^5 - 0).$$

   Second, we need to verify that this holds: $0^5 - 0 = 0 \Rightarrow 0$ is divisible by 5.
3. **State the Inductive Hypothesis:** We assume that $P(k)$ holds for all $k \geq 0$:

$$P(k) : 5|(k^5 - k).$$

4. **Inductive Step:** We show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers $k$. That is, we prove that $P(k + 1)$ must hold assuming that $P(k)$ holds for all positive integers $k$. Consider the statement $P(k)$:

$$5|(k^5 - k).$$

Therefore, $\exists u$ such that $k^5 - k = 5u$.

Note that:

$$(k + 1)^5 - (k + 1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k + 1)$$

$$= (k^5 + 1 - k - 1) + (5k^4 + 10k^3 + 10k^2 + 5k)$$

$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

Using our inductive hypothesis:

$$(k + 1)^5 = 5u + 5(k^4 + 2k^3 + 2k^2 + k)$$

Therefore, $(k + 1)^5$ is divisible by 5.

Therefore, $P(k+1)$ holds, so we have shown that $P(k) \rightarrow P(k+1)$. In conclusion, the statement holds by the Principle of Mathematical Induction.

• Rosen 4.2, Problem 3, page 291. (Similar to homework problem Rosen 4.2.4)

Using the Second Principle of Mathematical Induction (or Strong Induction), we will prove that:

$$\forall n \geq 8, \text{ we can form } n \text{ cents using just 3-cent stamps and 5-cent stamps.}$$

We complete the following four steps:

1. **State the propositional predicate:**

$$P(n) : n \text{ cents can be formed using just 3-cent stamps and 5-cent stamps.}$$

2. **Basis Step:** We verify that $P(8)$ holds. $P(8) :$ one 5-cent stamp and one 3-cent stamp.

3. **State the Inductive Hypothesis:** We assume that $P(j)$ holds for all $8 \leq j \leq k$.

4. **Inductive Step:** We show that the conditional statement $[P(1) \land P(2) \land \ldots \land P(k)] \rightarrow P(k + 1)$.

By the inductive hypothesis, $P(k-2)$ must hold. Therefore, $P(k+1)$ must hold because it requires the same amount of coins as $P(k-2)$, plus 1 more 3-cent stamp. Or, $P(k+1) : P(k-2)$ and one more 3-cent stamp. Therefore, $P(k+1)$ holds, so we have shown that $[P(1) \land P(2) \land \ldots \land P(k)] \rightarrow P(k+1)$. In conclusion, the statement holds by the Second Principle of Mathematical Induction.

• (Last 10 minutes) Quiz