Week 12 Recitation

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- Questions about lecture / homework so far?
- Questions from refresher sheet...
- Rosen 4.1, Problem 51, Page 281.

  A knight on a chessboard can move

  – One space horizontally (in either direction) and two spaces vertically (in either direction) or
  – Two spaces horizontally (in either direction) and one space vertically (in either direction).

  Suppose that we have an infinite chessboard, made up of all squares (i, j) where i and j are nonnegative integers. Use mathematical induction to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves. [*Hint*: Use induction on the variable $n = i + j$.]

  We want to prove that a knight can visit every square on the chessboard, using induction on the *city block distance* of the knight to a square. We complete the following four steps:

  1. *State the propositional predicate:*

     $$P(n)\text{: A knight can reach all squares } (i, j) \text{ where } i + j \leq n.$$

     Notice here that the propositional predicate is defined for n and is not prefixe by the quantifier $\forall n$.
  2. *Basis Step:* We must form and verify the basis case.

     First, we propose the following basis case:

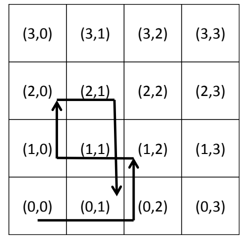
     $$P(2)\text{: } \forall i, j \in \mathbb{N} \text{ and } (i + j \leq 2)\text{, a knight can reach square } (i, j).$$

     Notice that the basis case corresponds to a knight being able to reach several squares, all those where $(i + j \leq 2)$.

     Second, we need to verify that $P(2)$ holds. We have 6 squares with $i + j \leq 2$:

(a) $(0,0)$: This state is reachable because this is where we start.

(b) $(0,1)$: The knight moves from (0,0) to (1,2) to (2,0) to (0,1).



(c) $(1,0)$: The knight moves from (0,0) to (2,1) to(0,2) to (1,0).

(d) $(2,0)$: The knight reached this square on its way to $(0,1)$.

(e) $(0,2)$: The knight reached this square on its way to $(1,0)$.

(f) $(1,1)$: The knight moves from (0,0) to (1,2) to (0,1) to (2,2) to (0,3) to (1,1).

3. *State the Inductive Hypothesis:* We assume that $P(k)$ holds for all $k \geq 2$:

$$P(k)\text{: A knight can reach all squares } (i,j) \text{ where } i+j \leq k.$$

4. *Inductive Step:* We show that the conditional statement $P(k) \to P(k+1)$ is true. That is, we prove that $P(k+1)$ must hold assuming that $P(k)$ holds. Consider the statement $P(k)$:

$$P(k)\text{: A knight can reach all squares } (i,j) \text{ where } i+j \leq k.$$

We want to show that a knight can reach all squares (i',j') where $i'+j' \leq k+1$. Because $k+1 \geq 3$, at least one of i' and j' can be chosen at least 2. WLOG, assume $i' \geq 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i'-2, j'+1)$, because $i'-2+j'+1 = i'+j'-1 = k$. From such a square, it is just one move to the square (i',j') Therefore, $P(k+1)$ holds, so we have shown that $P(k) \to P(k+1)$.

In conclusion, the statement holds by the Principle of Mathematical Induction. $\square$

- Rosen 4.1, Problem 5, Page 280. (Similar to homework problem Rosen 4.1.4)

  Using the Principle of Mathematical Induction, we will prove that

$$\forall n \in \mathbb{N}, 1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3.$$

We complete the following four steps:

1. *State the propositional predicate:*

$$P(n) : 1^2 + 3^2 + 5^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3.$$

2. *Basis Step:* We verify that $P(1)$ is true.
   We must form the basis step. First, we can write out what $P(1)$ is.
   $$P(1): 1^2 + 3^2 = (1+1)(2 \times 1 + 1)(2 \times 1 + 3)/3.$$
   Second, we need to verify that this holds: $1 + 9 = (2)(2+1)(2+3)/3 \Rightarrow 10 = 2 * 3 * 5 \Rightarrow 10 = 10$.
3. *State the Inductive Hypothesis:* We assume that $P(k)$ holds for all $k \geq 1$:
   $$P(k): 1^2 + 3^2 + 5^2 + \ldots + (2k+1)^2 = (k+1)(2k+1)(2k+3)/3.$$
4. *Inductive Step:* We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k. That is, we prove that $P(k+1)$ must hold assuming that $P(k)$ holds for all positive integers k. Consider the statement $P(k)$:
   $$1^2 + 3^2 + \ldots + (2k+1)^2 = (k+1)(2k+1)(2k+3)/3.$$
   Adding the term $(2k+3)^2$ on both sides of the equality, we get:
   $$\begin{aligned} 1^2 + 3^2 + \ldots + (2k+1)^2 + (2k+3)^3 &= (k+1)(2k+1)(2k+3)/3 + (2k+3)^2 \\ &= (2k+3)[(k+1)(2k+1)/3 + (2k+3)] \\ &= (2k+3)[(2k^2+3k+1)/3 + (6k+9)/3] \\ &= (2k+3)[(2k^2+9k+10)/3] \\ &= (2k+3)(2k+5)(k+2)/3 \\ &= ([k+1]+1)(2[k+1]+1)(2[k+1]+3)/3 \end{aligned}$$
   Therefore, $P(k+1)$ holds, so we have shown that $P(k) \to P(k+1)$. In conclusion, the statement holds by the Principle of Mathematical Induction. $\square$

- Rosen 4.1, Problem 33, page 280. (Similar to homework problem Rosen 4.1.32)

  Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

  We complete the following four steps:

  1. *State the propositional predicate:*
     $$P(n): 5|(n^5 - n).$$
  2. *Basis Step:* We verify that $P(1)$ is true.
     We must form the basis step. First, we can write out what $P(0)$ is.
     $$P(0): 5|(0^5 - 0).$$
     Second, we need to verify that this holds: $0^5 - 0 = 0 \Rightarrow 0$ is divisible by 5.

3. *State the Inductive Hypothesis:* We assume that $P(k)$ holds for all $k \geq 0$:

$$P(k) : 5|(k^5 - k).$$

4. *Inductive Step:* We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k. That is, we prove that $P(k+1)$ must hold assuming that $P(k)$ holds for all positive integers k. Consider the statement $P(k)$:

$$5|(k^5 - k).$$

Therefore, $\exists u$ such that $k^5 - k = 5u$.
Note that:

$$\begin{aligned} (k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1) \\ &= (k^5 + 1 - k - 1) + (5k^4 + 10k^3 + 10k^2 + 5k) \\ &= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k) \end{aligned}$$

Using our inductive hypothesis:

$$(k+1)^5 = 5u + 5(k^4 + 2k^3 + 2k^2 + k)$$

Therefore, $(k+1)^5$ is divisible by 5.
Therefore, $P(k+1)$ holds, so we have shown that $P(k) \to P(k+1)$. In conclusion, the statement holds by the Principle of Mathematical Induction. $\square$

- Rosen 4.2, Problem 3, page 291. (Similar to homework problem Rosen 4.2.4)

Using the Second Principle of Mathematical Induction (or *Strong* Induction), we will prove that:

$\forall n \geq 8$, we can form n cents using just 3-cent stamps and 5-cent stamps.

We complete the following four steps:

1. *State the propositional predicate:*

$P(n)$: n cents can be formed using just 3-cent stamps and 5-cent stamps.

2. *Basis Step:* We verify that $P(8)$ holds. $P(8)$: one 5-cent stamp and one 3-cent stamp.
3. *State the Inductive Hypothesis:* We assume that $P(j)$ holds for all $8 \leq j \leq k$.
4. *Inductive Step:* We show that the conditional statement $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \to P(k+1)$.

By the inductive hypothesis, $P(k-2)$ must hold. Therefore, $P(k+1)$ must hold because it requires the same amount of coins as $P(k-2)$, plus 1 more 3-cent stamp. Or, $P(k+1) : P(k-2)$ and one more 3-cent stamp. Therefore, $P(k+1)$ holds, so we have shown that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \to P(k+1)$. In conclusion, the statement holds by the Second Principle of Mathematical Induction. $\square$

- (Last 10 minutes) Quiz