Sets

Sections 2.1 and 2.2 of Rosen

Spring 2011

CSCE 235 Introduction to Discrete Structures

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Outline

- Definitions: set, element
- Terminology and notation
  - Set equal, multi-set, bag, set builder, intension, extension, Venn Diagram (representation), empty set, singleton set, subset, proper subset, finite/infinite set, cardinality
- Proving equivalences
- Power set
- Tuples (ordered pair)
- Cartesian Product (a.k.a. Cross product), relation
- Quantifiers
- Set Operations (union, intersection, complement, difference), Disjoint sets
- Set equivalences (cheat sheet or Table 1, page 124)
  - Inclusion in both directions
  - Using membership tables
- Generalized Unions and Intersection
- Computer Representation of Sets

Introduction (1)

- We have already implicitly dealt with sets
  - Integers (*Z*), rationals (*Q*), naturals (*N*), reals (*R*), etc.
- We will develop more fully
  - The definitions of sets
  - The properties of sets
  - The operations on sets
- **Definition**: A set is an <u>unordered</u> collection of (<u>unique</u>) objects
- Sets are fundamental discrete structures and for the basis of more complex discrete structures like graphs

Introduction (2)

- **Definition**: The objects in a set are called elements or members of a set. A set is said to contain its elements
- Notation, for a set A:
  - $x \in A$: x is an element of A $\in$
  - $x \notin A$: x is not an element of A $\notin$

Terminology (1)

- **Definition**: Two sets, A and B, are equal is they contain the same elements. We write A=B.
- Example:
  - {2,3,5,7}={3,2,7,5}, because a set is unordered
  - Also, {2,3,5,7}={2,2,3,5,3,7} because a set contains unique elements
  - However, {2,3,5,7} $\neq${2,3} $\neq

Terminology (2)

- A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where
  - m_1 occurs a_1 times
  - m_2 occurs a_2 times
  - …
  - m_r occurs a_r times
- In Databases, we distinguish
  - A set: elements cannot be repeated
  - A bag: elements can be repeated

Terminology (3)

- The **set-builder** notation

  $$O=\{ x \mid (x \in \mathcal{Z}) \wedge (x=2k) \text{ for some } k \in Z\}$$

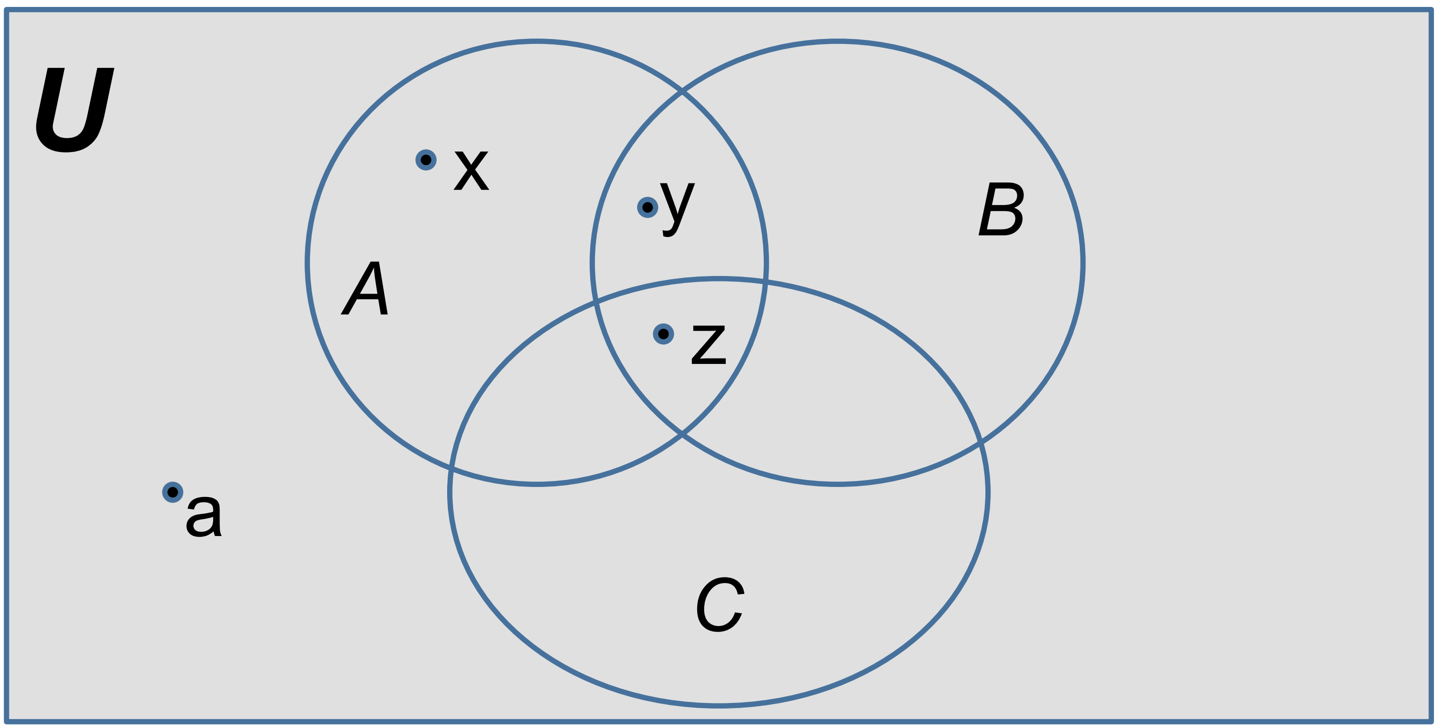
  reads: O is the set that contains all x such that x is an integer and x is even
- A set is defined in **intension** when you give its set-builder notation

  $$O=\{ x \mid (x \in \mathcal{Z}) \wedge (0 \leq x \leq 8) \wedge (x=2k) \text{ for some } k \in Z \}$$
- A set is defined in **extension** when you enumerate all the elements:

  $$O=\{0,2,4,6,8\}$$

Venn Diagram: Example

- A set can be represented graphically using a Venn Diagram



More Terminology and Notation (1)

- A set that has no elements is called the **empty set** or **null set** and is denoted $\emptyset$ $\$\backslash$emptyset$\$$
- A set that has one element is called a **singleton set**.
  - For example: {a}, with brackets, is a singleton set
  - a, without brackets, is an element of the set {a}
- Note the subtlety in $\emptyset \neq \{\emptyset\}$
  - The left-hand side is the empty set
  - The right hand-side is a singleton set, and a set containing a set

More Terminology and Notation (2)

- **Definition**: A is said to be a **subset** of B, and we write $A \subseteq B$, if and only if every element of A is also an element of B $\subseteq$
- That is, we have the equivalence:

$$A \subseteq B \Leftrightarrow \forall x\,(x \in A \Rightarrow x \in B)$$

More Terminology and Notation (3)

- **Theorem**: For any set S *Theorem 1, page 115*
  - $\emptyset \subseteq S$ and
  - $S \subseteq S$
- The proof is in the book, an excellent example of a vacuous proof

More Terminology and Notation (4)

- **Definition**: A set A that is a subset of a set B is called a **proper subset** if $A \neq B$.
- That is there is an element $x \in B$ such that $x \notin A$
- We write: $A \subset B$, $A \subsetneq B$
- In LaTex: $\subset$, $\subsetneq$

More Terminology and Notation (5)

- Sets can be elements of other sets
- Examples
  - $S_1 = \{\emptyset,\{a\},\{b\},\{a,b\},c\}$
  - $S_2=\{\{1\},\{2,4,8\},\{3\},\{6\},4,5,6\}$

More Terminology and Notation (6)

- **Definition**: If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that:
  - S is a finite set, and
  - The cardinality of S is n. Notation: $|S| = n$.
- **Definition**: A set that is not finite is said to be infinite

More Terminology and Notation (7)

- Examples
  - Let B = {x | ($x \leq 100$) $\wedge$ (x is prime)}, the cardinality of B is $|B|=25$ because there are 25 primes less than or equal to 100.
  - The cardinality of the empty set is $|\varnothing|=0$
  - The sets *N*, *Z*, *Q*, *R* are all infinite

Proving Equivalence (1)

- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x (x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
  - Any proof method can be used.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x\, (x \in B) \wedge (x \notin A)$

Proving Equivalence (2)

- Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

we will see an example later..

Power Set (1)

- **Definition**: The power set of a set S, denoted P (S), is the set of all subsets of S.
- Examples
  - Let A={a,b,c}, P(A)={∅,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
  - Let A={{a,b},c}, P(A)={∅,{{a,b}},{c},{{a,b},c}}
- Note: the empty set ∅ and the set itself are always elements of the power set. This fact follows from Theorem 1 (Rosen, page 115).

Power Set (2)

- The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set
- **Fact**: Let S be a set such that |S|=n, then

$$|P(S)| = 2^n$$

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- Proving equivalences
- Power set
- **Tuples (ordered pair)**
- **Cartesian Product (a.k.a. Cross product), relation**
- **Quantifiers**
- Set Operations (union, intersection, complement, difference), Disjoint sets
- Set equivalences (cheat sheet or Table 1, page 124)
  - Inclusion in both directions
  - Using membership tables
- Generalized Unions and Intersection
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Tuples (1)

- Sometimes we need to consider ordered collections of objects
- **Definition**: The ordered n-tuple $(a_1,a_2,\ldots,a_n)$ is the ordered collection with the element a_i being the i-th element for $i=1,2,\ldots,n$
- Two ordered n-tuples $(a_1,a_2,\ldots,a_n)$ and $(b_1,b_2,\ldots,b_n)$ are equal iff for every $i=1,2,\ldots,n$ we have $a_i=b_i$ $(a_1,a_2,\ldots,a_n)$
- A 2-tuple (n=2) is called an ordered pair

Cartesian Product (1)

- **Definition**: Let A and B be two sets. The Cartesian product of A and B, denoted AxB, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{ (a,b) \mid (a \in A) \land (b \in B) \}$$

- The Cartesian product is also known as the cross product
- **Definition**: A subset of a Cartesian product, $R \subseteq A \times B$ is called a relation. We will talk more about relations in the next set of slides
- Note: $A \times B \neq B \times A$ unless $A=\emptyset$ or $B=\emptyset$ or $A=B$. Find a counter example to prove this.

Cartesian Product (2)

- Cartesian Products can be generalized for any n-tuple
- **Definition**: The Cartesian product of n sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is

  $A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i=1,2,\ldots,n\}$

Notation with Quantifiers

- Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language
- Now we can simplify things using <u>set notation</u>!
- Example
  - $\forall x \in R\ (x^2 \geq 0)$
  - $\exists x \in Z\ (x^2 = 1)$
  - Also mixing quantifiers:

$$\forall a,b,c \in R\ \exists x \in C\ (ax^2+bx+c=0)$$

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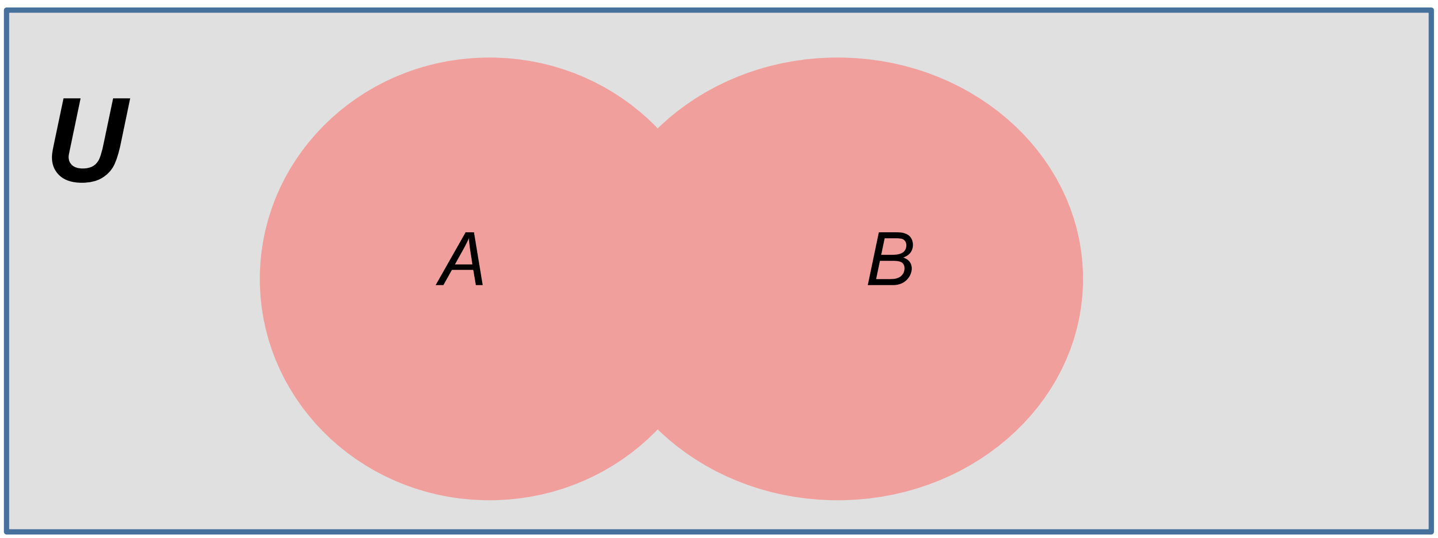
Set Operations

- Arithmetic operators (+,-, × ,÷) can be used on pairs of numbers to give us new numbers
- Similarly, set operators exist and act on two sets to give us new sets
  - Union $\cup$
  - Intersection $\cap$
  - Set difference $\setminus$
  - Set complement $\overline{S}$
  - Generalized union $\bigcup$
  - Generalized intersection $\bigcap$

Set Operators: Union

- **Definition**: The union of two sets A and B is the set that contains all elements in A, B, or both. We write:

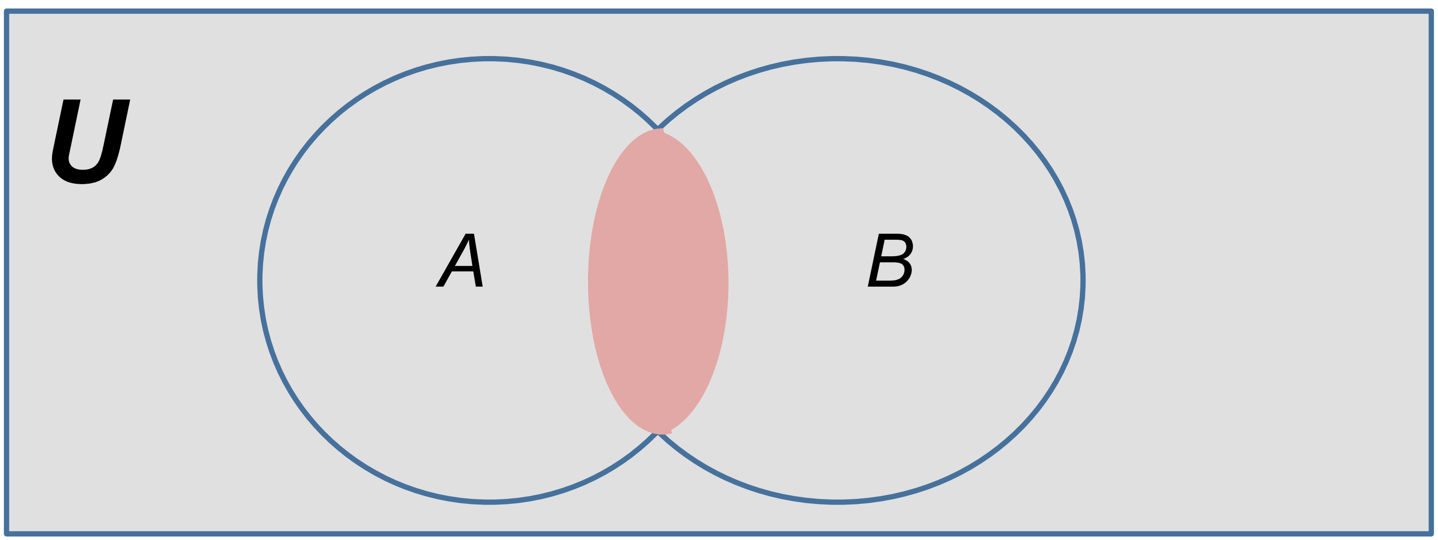
$$A \cup B = \{ x \mid (x \in A) \vee (x \in B) \}$$



Set Operators: Intersection

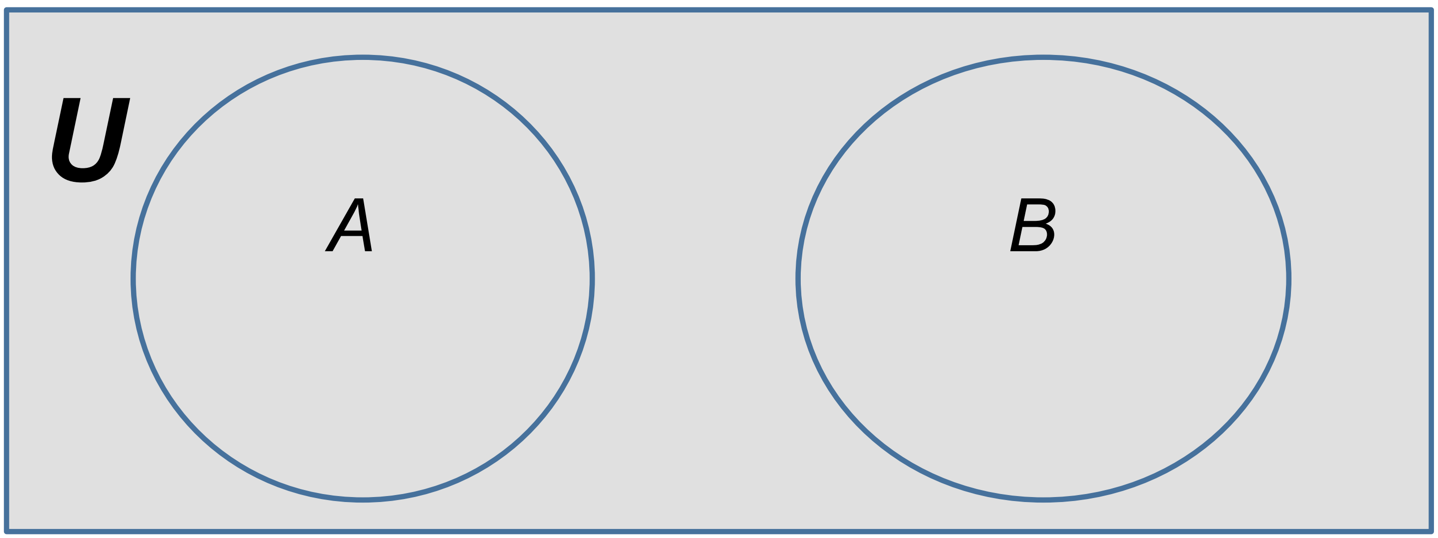
- **Definition**: The intersection of two sets A and B is the set that contains all elements that are element of both A and B. We write:

$$A \cap B = \{ x \mid (x \in A) \wedge (x \in B) \}$$



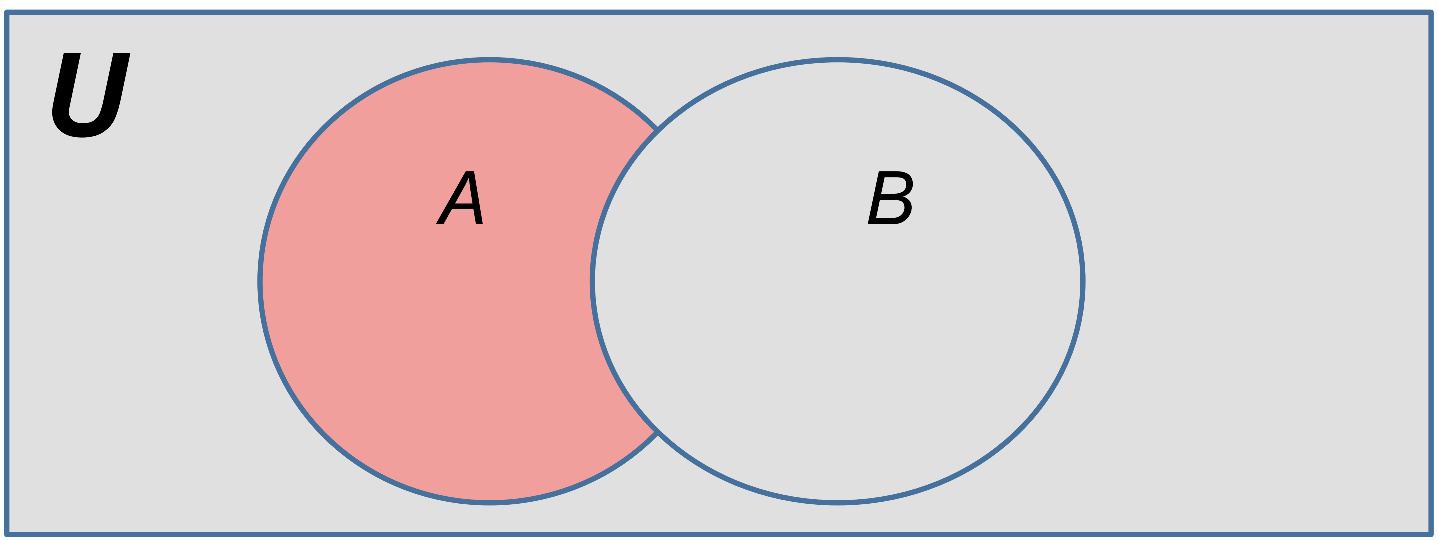
Disjoint Sets

- **Definition**: Two sets are said to be disjoint if their intersection is the empty set: $A \cap B = \varnothing$



Set Difference

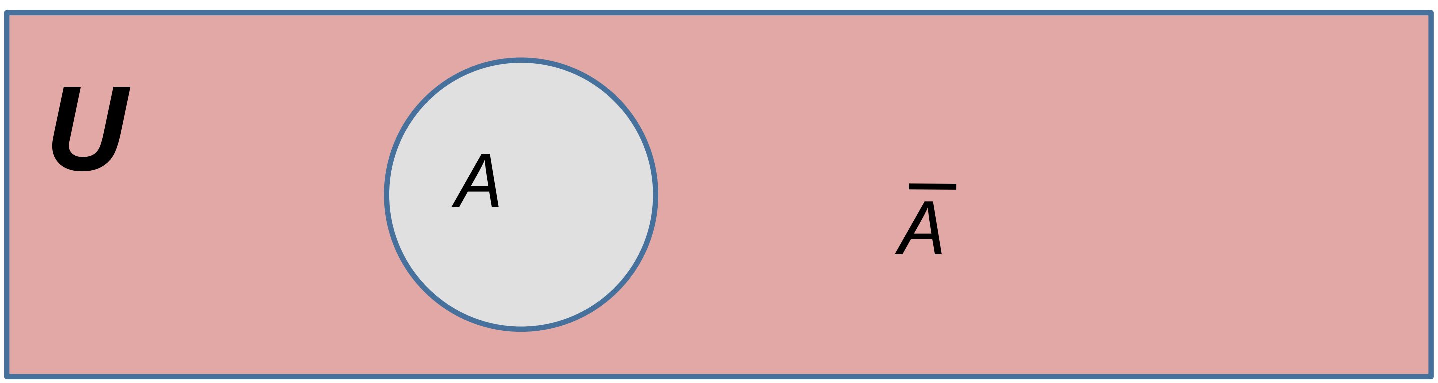
- **Definition**: The difference of two sets A and B, denoted A\B ($\setminus$) or A−B, is the set containing those elements that are in A but not in B



Set Complement

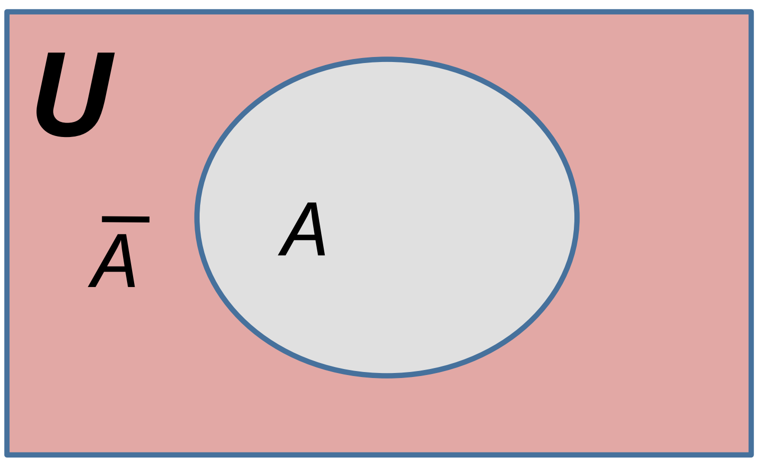
- **Definition**: The complement of a set A, denoted $\bar{A}$ ($\bar$), consists of all elements not in A. That is the difference of the universal set and U: U\A

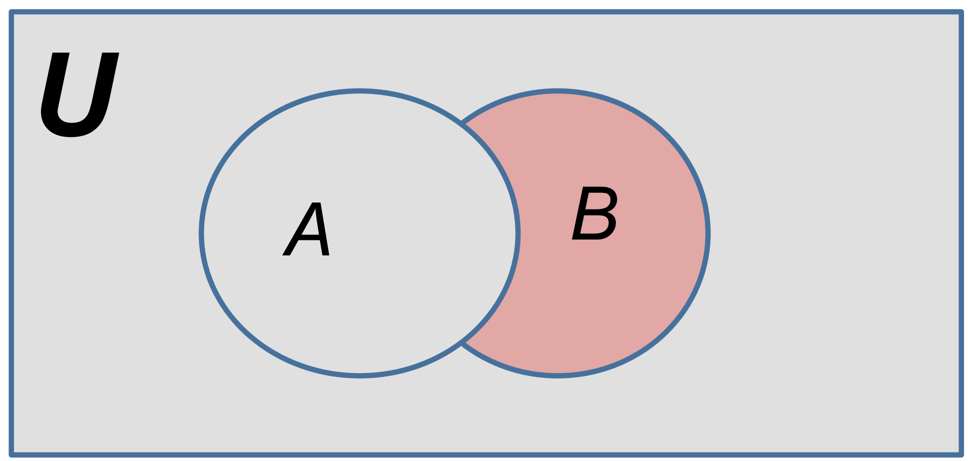
$$\bar{A} = A^C = \{x \mid x \notin A\}$$



Set Complement: Absolute & Relative

- Given the Universe U, and A,B ⊂ U.
- The (absolute) complement of A is A=U\A
- The (relative) complement of A in B is B\A





Set Idendities

- There are analogs of all the usual laws for set operations. Again, the Cheat Sheat is available on the course webpage: http://www.cse.unl.edu/~cse235/files/LogicalEquivalences.pdf
- Let's take a quick look at this Cheat Sheet or at Table 1 on page 124 in your textbook

Proving Set Equivalences

- Recall that to prove such identity, we must show that:
  1. The left-hand side is a subset of the right-hand side
  2. The right-hand side is a subset of the left-hand side
  3. Then conclude that the two sides are thus equal
- The book proves several of the standard set identities
- We will give a couple of different examples here

Proving Set Equivalences: Example A (1)

- Let
  - $A=\{x|x \text{ is even}\}$
  - $B=\{x|x \text{ is a multiple of } 3\}$
  - $C=\{x|x \text{ is a multiple of } 6\}$
- Show that $A\cap B=C$

Proving Set Equivalences: Example A (2)

- **$A \cap B \subseteq C$**: $\forall x \in A \cap B$

  $\Rightarrow$ x is a multiple of 2 and x is a multiple of 3

  $\Rightarrow$ we can write x=2.3.k for some integer k

  $\Rightarrow$ x=6k for some integer k $\Rightarrow$ x is a multiple of 6

  $\Rightarrow x \in C$

- **$C \subseteq A \cap B$**: $\forall x \in C$

  $\Rightarrow$ x is a multiple of 6 $\Rightarrow$ x=6k for some integer k

  $\Rightarrow$ x=2(3k)=3(2k) $\Rightarrow$ x is a multiple of 2 and of 3

  $\Rightarrow x \in A \cap B$

Proving Set Equivalences: Example B (1)

- An alternative prove is to use membership tables where an entry is
  - 1 if a chosen (but fixed) element is in the set
  - 0 otherwise
- Example: Show that

$$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$$

Proving Set Equivalences: Example B (2)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
0	0	0	0	**1**	1	1	1	**1**
0	0	1	0	**1**	1	1	0	**1**
0	1	0	0	**1**	1	0	1	**1**
0	1	1	0	**1**	1	0	0	**1**
1	0	0	0	**1**	0	1	1	**1**
1	0	1	0	**1**	0	1	0	**1**
1	1	0	0	**1**	0	0	1	**1**
1	1	1	1	**0**	0	0	0	**0**

- 1 under a set indicates that “an element is in the set”
- If the columns are equivalent, we can conclude that indeed the two sets are equal

Generalizing Set Operations: Union and Intersection

- In the previous example, we showed De Morgan's Law generalized to unions involving 3 sets
- In fact, De Morgan's Laws hold for any finite set of sets
- Moreover, we can generalize set operations union and intersection in a straightforward manner to any finite number of sets

Generalized Union

- **Definition**: The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

LaTeX: $\Bigcup_{i=1}^{n}A_i=A_1\cup A_2 \cup\ldots\cup A_n$

Generalized Intersection

- **Definition**: The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

LaTex: $\Bigcap_{i=1}^{n}A_i=A_1\cap A_2 \cap\ldots\cap A_n$

Computer Representation of Sets (1)

- There really aren't ways to represent infinite sets by a computer since a computer has a finite amount of memory
- If we assume that the universal set U is finite, then we can easily and effectively represent sets by bit vectors
- Specifically, we force an ordering on the objects, say:

$$U=\{a_1, a_2,\ldots,a_n\}$$

- For a set $A \subseteq U$, a bit vector can be defined as, for i=1,2,…,n
  - $b_i=0$ if $a_i \notin A$
  - $b_i=1$ if $a_i \in A$

Computer Representation of Sets (2)

- Examples
  - Let U={0,1,2,3,4,5,6,7} and A={0,1,6,7}
  - The bit vector representing A is: 1100 0011
  - How is the empty set represented?
  - How is U represented?
- Set operations become trivial when sets are represented by bit vectors
  - Union is obtained by making the bit-wise OR
  - Intersection is obtained by making the bit-wise AND

Computer Representation of Sets (3)

- Let U={0,1,2,3,4,5,6,7}, A={0,1,6,7}, B={0,4,5}
- What is the bit-vector representation of B?
- Compute, bit-wise, the bit-vector representation of $A \cap B$
- Compute, bit-wise, the bit-vector representation of $A \cup B$
- What sets do these bit vectors represent?

Programming Question

- Using bit vector, we can represent sets of cardinality equal to the size of the vector
- What if we want to represent an arbitrary sized set in a computer (i.e., that we do not know a priori the size of the set)?
- What data structure could we use?