

Sequences & Summations

Section 2.4 of Rosen

Spring 2011

CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Questions: cse235@cse.unl.edu

Outline

Although you are (more or less) familiar with sequences and summations, we give a quick review

- Sequences
 - Definition, 2 examples
- Progressions: Special sequences
 - Geometric, arithmetic
- Summations
 - Careful when changing lower/upper limits
- Series: Sum of the elements of a sequence
 - Examples, infinite series, convergence of a geometric series

Sequences

- **Definition:** A sequence is a function from a subset of integers to a set S . We use the notation(s):

$$\{a_n\} \quad \{a_n\}_n^\infty \quad \{a_n\}_{n=0}^\infty$$

- Each a_n is called the n^{th} term of the sequence
- We rely on the context to distinguish between a sequence and a set, although they are distinct structures

Sequences: Example 1

- Consider the sequence

$$\left\{ \left(1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty}$$

- The terms of the sequence are:

$$a_1 = \left(1 + \frac{1}{1} \right)^1 = 2.00000$$

$$a_2 = \left(1 + \frac{1}{2} \right)^2 = 2.25000$$

$$a_3 = \left(1 + \frac{1}{3} \right)^3 = 2.37037$$

$$a_4 = \left(1 + \frac{1}{4} \right)^4 = 2.44140$$

$$a_5 = \left(1 + \frac{1}{5} \right)^5 = 2.48832$$

- What is this sequence?
- The sequence corresponds to $\lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty} = e = 2.71828..$

Sequences: Example 2

- The sequence: $\{h_n\}_{n=1}^{\infty} = 1/n$
is known as the harmonic sequence

- The sequence is simply:

$$1, 1/2, 1/3, 1/4, 1/5, \dots$$

- This sequence is particularly interesting because its summation is divergent:

$$\sum_{n=1}^{\infty} (1/n) = \infty$$

Progressions: Geometric

- **Definition:** A geometric progression is a sequence of the form

$$a, ar, ar^2, ar^3, \dots, ar^n, \dots$$

Where:

- $a \in \mathbb{R}$ is called the initial term
- $r \in \mathbb{R}$ is called the common ratio
- A geometric progression is a discrete analogue of the exponential function

$$f(x) = ar^x$$

Geometric Progressions: Examples

- A common geometric progression in Computer Science is:

$$\{a_n\} = 1/2^n$$

with $a=1$ and $r=1/2$

- Give the initial term and the common ratio of
 - $\{b_n\}$ with $b_n = (-1)^n$
 - $\{c_n\}$ with $c_n = 2(5)^n$
 - $\{d_n\}$ with $d_n = 6(1/3)^n$

Progressions: Arithmetic

- **Definition:** An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, a+3d, \dots, a+nd, \dots$$

Where:

- $a \in R$ is called the initial term
 - $d \in R$ is called the common difference
- An arithmetic progression is a discrete analogue of the linear function

$$f(x) = dx+a$$

Arithmetic Progressions: Examples

- Give the initial term and the common difference of
 - $\{s_n\}$ with $s_n = -1 + 4n$
 - $\{t_n\}$ with $s_n = 7 - 3n$

More Examples

- Table 1 on Page 153 (Rosen) has some useful sequences:

$$\{n^2\}_{n=1}^{\infty}, \{n^3\}_{n=1}^{\infty}, \{n^4\}_{n=1}^{\infty}, \{2^n\}_{n=1}^{\infty}, \{3^n\}_{n=1}^{\infty}, \{n!\}_{n=1}^{\infty}$$

Outline

Although you are (more or less) familiar with sequences and summations, we give a quick review

- Sequences
 - Definition, 2 examples
- Progressions: Special sequences
 - Geometric, arithmetic
- **Summations**
 - **Careful when changing lower/upper limits**
- Series: Sum of the elements of a sequence
 - Examples, infinite series, convergence of a geometric series

Summations (1)

- You should be by now familiar with the summation notation:

$$\sum_{j=m}^n (a_j) = a_m + a_{m+1} + \dots + a_{n-1} + a_n$$

Here

- j is the index of the summation
 - m is the lower limit
 - n is the upper limit
- Often times, it is useful to change the lower/upper limits, which can be done in a straightforward manner (although we must be very careful):

$$\sum_{j=1}^n (a_j) = \sum_{i=0}^{n-1} (a_{i+1})$$

Summations (2)

- Sometimes we can express a summation in closed form, as for geometric series
- **Theorem:** For $a, r \in \mathbb{R}, r \neq 0$

$$\sum_{i=0}^n (ar^i) = \begin{cases} (ar^{n+1}-a)/(r-1) & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Closed form = analytical expression using a bounded number of well-known functions, does not involved an infinite series or use of recursion

Summations (3)

- Double summations often arise when analyzing an algorithm

$$\sum_{i=1}^n \sum_{j=1}^i (a_j) = a_1 +$$

$$a_1 + a_2 +$$

$$a_1 + a_2 + a_3 +$$

...

$$a_1 + a_2 + a_3 + \dots + a_n$$

- Summations can also be indexed over elements in a set:

$$\sum_{s \in S} f(s)$$

- Table 2 on Page 157 (Rosen) has very useful summations. Exercises 2.4.14—18 are **great** material to practice on.

Outline

Although you are (more or less) familiar with sequences and summations, we give a quick review

- Sequences
 - Definition, 2 examples
- Progressions: Special sequences
 - Geometric, arithmetic
- Summations
 - Careful when changing lower/upper limits
- **Series: Sum of the elements of a sequence**
 - **Examples, infinite series, convergence of a geometric series**

Series

- When we take the sum of a sequence, we get a series
- We have already seen a closed form for geometric series
- Some other useful closed forms include the following:
 - $\sum_{i=k}^u 1 = u-k+1$, for $k \leq u$
 - $\sum_{i=0}^n i = n(n+1)/2$
 - $\sum_{i=0}^n (i^2) = n(n+1)(2n+1)/6$
 - $\sum_{i=0}^n (i^k) \approx n^{k+1}/(k+1)$

Infinite Series

- Although we will mostly deal with finite series (i.e., an upper limit of n for fixed integer), infinite series are also useful
- Consider the following geometric series:
 - $\sum_{n=0}^{\infty} (1/2^n) = 1 + 1/2 + 1/4 + 1/8 + \dots$ converges to 2
 - $\sum_{n=0}^{\infty} (2^n) = 1 + 2 + 4 + 8 + \dots$ does not converge
- However note: $\sum_{n=0}^n (2^n) = 2^{n+1} - 1$ ($a=1, r=2$)

Infinite Series: Geometric Series

- In fact, we can generalize that fact as follows
- **Lemma:** A geometric series converges if and only if the absolute value of the common ratio is less than 1

When $|r| < 1$,

$$\begin{aligned}\lim_{n \rightarrow \infty} \sum_{i=0}^n (ar^i) &= \lim_{n \rightarrow \infty} \sum_{i=0}^n (ar^{n+1} - a)/(r-1) \\ &= a/(1-r)\end{aligned}$$