Partial Orders

Section 8.6 of Rosen
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CSCE 235 Introduction to Discrete Structures
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Outline

• Motivating example
• Definitions
  – Partial ordering, comparability, total ordering, well ordering
• Principle of well-ordered induction
• Lexicographic orderings
• Hasse Diagrams
• Extremal elements
• Lattices
• Topological Sorting
Motivating Example (1)

• Consider the renovation of Avery Hall. In this process several tasks were undertaken
  – Remove Asbestos
  – Replace windows
  – Paint walls
  – Refinish floors
  – Assign offices
  – Move in office furniture
Motivating Example (2)

• Clearly, some things had to be done before others could begin
  – Asbestos had to be removed before anything (except assigning offices)
  – Painting walls had to be done before refinishing floors to avoid ruining them, etc.

• On the other hand, several things could be done concurrently:
  – Painting could be done while replacing the windows
  – Assigning offices could be done at anytime before moving in office furniture

• This scenario can be nicely modeled using partial orderings
Partial Orderings: Definitions

• **Definitions:**
  
  – A relation $R$ on a set $S$ is called a **partial order** if it is
    • Reflexive
    • Antisymmetric
    • Transitive
  
  – A set $S$ together with a partial ordering $R$ is called a **partially ordered set** (poset, for short) and is denote $(S,R)$
  
• Partial orderings are used to give an order to sets that may not have a natural one

• In our renovation example, we could define an ordering such that $(a,b) \in R$ if ‘$a$ must be done before $b$ can be done’
Partial Orderings: Notation

• We use the notation:
  – $a \preccurlyeq b$, when $(a,b) \in R$
  – $a \prec b$, when $(a,b) \in R$ and $a \neq b$

• The notation $\preceq$ is not to be mistaken for “less than” ($\preceq$ versus $\leq$)

• The notation $\preceq$ is used to denote any partial ordering
Comparability: Definition

• **Definition:**
  
  – The elements $a$ and $b$ of a poset $(S, \preceq)$ are called **comparable** if either $a \preceq b$ or $b \preceq a$.
  
  – When for $a, b \in S$, we have neither $a \preceq b$ nor $b \preceq a$, we say that $a, b$ are **incomparable**

• Consider again our renovation example
  
  – Remove Asbestos $\prec a_i$ for all activities $a_i$ except assign offices
  
  – Paint walls $\prec$ Refinish floors
  
  – Some tasks are incomparable: Replacing windows can be done before, after, or during the assignment of offices
Total orders: Definition

• **Definition:**
  – If \((S,\preceq)\) is a poset and every two elements of \(S\) are comparable, \(S\) is called a **totally ordered set**.
  – The relation \(\preceq\) is said to be a **total order**

• **Example**
  – The relation “less than or equal to” over the set of integers \((\mathbb{Z}, \leq)\) since for every \(a,b\in\mathbb{Z}\), it must be the case that \(a\leq b\) or \(b\leq a\)
  – What happens if we replace \(\leq\) with \(<\)?

  The relation \(<\) is not reflexive, and \((\mathbb{Z},<)\) is not a poset
Well Orderings: Definition

- **Definition**: $(S, \preceq)$ is a well-ordered set if
  - It is a poset
  - Such that $\preceq$ is a total ordering and
  - Such that every non-empty subset of $S$ has a least element

- **Example**
  - The natural numbers along with $\leq$, $(N, \leq)$, is a well-ordered set since any nonempty subset of $N$ has a least element and $\leq$ is a total ordering on $N$
  - However, $(\mathbb{Z}, \leq)$ is not a well-ordered set
    - Why? $\mathbb{Z} \not\subseteq \mathbb{Z}$ but does not have a least element
    - Is it totally ordered? Yes
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Principle of Well-Ordered Induction

- Well-ordered sets are the basis of the proof technique known as induction (more when we cover Chapter 3)
- **Theorem: Principle of Well-Ordered Induction**
  
  Given $S$ is a well-ordered set. $P(x)$ is true for all $x \in S$ if
  
  (Basis Step: $P(x_0)$ is true for the least element in $S$ and)
  
  Inductive Step: For every $y \in S$ if $P(x)$ is true for all $x < y$, then $P(y)$ is true
Principle of Well-Ordered Induction: Proof

**Proof:** (S well ordered) \(\land\) (Basis Step) \(\land\) (Induction Step) \(\Rightarrow\) \(\forall x \in S, P(x)\)

- Suppose that it is not the case the \(P(x)\) holds for all \(x \in S\)
  \[\Rightarrow \exists y \ P(y) \text{ is false}\]
  \[\Rightarrow A = \{ x \in S \mid P(x) \text{ is false} \} \text{ is not empty}\]
- \(S\) is well ordered \(\Rightarrow A\) has a least element \(a\)
- Since \(P(x_0)\) is true and \(P(a)\) is false \(\Rightarrow a \neq x_0\)
- \(P(x)\) holds for all \(x \in S\) and \(x \prec a\), then \(P(a)\) holds by the induction step
- This yields a contradiction  \(\text{QED}\)
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• **Lexicographic orderings**
  – Idea, on $A_1 \times A_2$, $A_1 \times A_2 \times \ldots \times A_n$, $S^t$ (strings)
• Hasse Diagrams
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Lexicographic Orderings: Idea

• Lexigraphic ordering is the same as any dictionary or phone-book ordering:
  – We use alphabetic ordering
    • Starting with the first character in the string
    • Then the next character, if the first was equal, etc.
  – If a word is shorter than the other, than we consider that the ‘no character’ of the shorter word to be less than ‘a’
Lexicographic Orderings on $A_1 \times A_2$

- Formally, lexicographic ordering is defined by two other orderings.
- **Definition:** Let $(A_1, \prec_1)$ and $(A_2, \prec_2)$ be two posets. The lexicographic ordering $\prec$ on the Cartesian product $A_1 \times A_2$ is defined by
  \[(a_1, a_2) \prec (a'_1, a'_2) \text{ if } (a_1 \prec_1 a'_1) \text{ or } (a_1 = a'_1 \text{ and } a_2 \prec_2 a'_2)\]
- If we add equality to the lexicographic ordering $\prec$ on $A_1 \times A_2$, we obtain a partial ordering.
Lexicographic Ordering on $A_1 \times A_2 \times \ldots \times A_n$

• Lexicographic ordering generalizes to the Cartesian Product of $n$ set in a natural way

• Define $\prec$ on $A_1 \times A_2 \times \ldots \times A_n$ by

$$ (a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n) $$

If $a_1 \prec b_1$ or of there is an integer $i>0$ such that

$$ a_1 = b_1, \ a_2 = b_2, \ldots, \ a_i = b_i \text{ and } a_{i+1} \prec b_{i+1} $$
Lexicographic Ordering on Strings

• Consider the two non-equal strings \(a_1a_2...a_m\) and \(b_1b_2...b_n\) on a poset \((S^t, \prec)\)

• Let
  – \(t=\min(n,m)\)
  – \(\prec\) be the lexicographic ordering on \(S^t\)

• \(a_1a_2...a_m\) is less than \(b_1b_2...b_n\) if and only if
  – \((a_1,a_2,...,a_t) \prec (b_1,b_2,...,b_t)\) or
  – \((a_1,a_2,...,a_t)=(b_1,b_2,...,b_t)\) and \(m<n\)
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Hasse Diagrams

• Like relations and functions, partial orders have a convenient graphical representation: Hasse Diagrams
  – Consider the digraph representation of a partial order
  – Because we are dealing with a partial order, we know that the relation must be reflexive and transitive
  – Thus, we can simplify the graph as follows
    • Remove all self loops
    • Remove all transitive edges
    • Remove directions on edges assuming that they are oriented upwards
  – The resulting diagram is far simpler
Hasse Diagram: Example
Hasse Diagrams: Example (1)

• Of course, you need not always start with the complete relation in the partial order and then trim everything.

• Rather, you can build a Hasse Diagram directly from the partial order.

• Example: Draw the Hasse Diagram
  – for the following partial ordering: \((a,b) \mid a \mid b\)
  – on the set \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}
  – (these are the divisors of 60 which form the basis of the ancient Babylonian base-60 numeral system)
Hasse Diagram: Example (2)
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Extremal Elements: Summary

We will define the following terms:

• A maximal/minimal element in a poset \((S, \prec)\)
• The maximum (greatest)/minimum (least) element of a poset \((S, \prec)\)
• An upper/lower bound element of a subset \(A\) of a poset \((S, \prec)\)
• The greatest lower/least upper bound element of a subset \(A\) of a poset \((S, \prec)\)
Extremal Elements: Maximal

• **Definition**: An element $a$ in a poset $(S, \preceq)$ is called **maximal** if it is not less than any other element in $S$. That is: $\neg (\exists b \in S (a \prec b))$

• If there is one **unique** maximal element $a$, we call it the **maximum** element (or the **greatest** element)
Extremal Elements: Minimal

• **Definition:** An element $a$ in a poset $(S, \preceq)$ is called **minimal** if it is not greater than any other element in $S$. That is: $\neg (\exists b \in S \ (b \preceq a))$

• If there is one **unique** minimal element $a$, we call it the **minimum** element (or the **least** element)
Extremal Elements: Upper Bound

• **Definition:** Let \((S, \preceq)\) be a poset and let \(A \subseteq S\). If \(u\) is an element of \(S\) such that \(a \preceq u\) for all \(a \in A\) then \(u\) is an **upper bound of** \(A\).

• An element \(x\) that is an upper bound on a subset \(A\) and is less than all other upper bounds on \(A\) is called the **least upper bound on** \(A\). We abbreviate it as \(\text{lub}\).
Extremal Elements: Lower Bound

- **Definition:** Let \((S, \preceq)\) be a poset and let \(A \subseteq S\). If \(l\) is an element of \(S\) such that \(l \preceq a\) for all \(a \in A\) then \(l\) is an **lower bound** of \(A\).

- An element \(x\) that is a lower bound on a subset \(A\) and is greater than all other lower bounds on \(A\) is called the **greatest lower bound** on \(A\). We abbreviate it \(\text{glb}\).
Extremal Elements: Example 1

What are the minimal, maximal, minimum, maximum elements?

- Minimal: \{a,b\}
- Maximal: \{c,d\}
- There are no unique minimal or maximal elements, thus no minimum or maximum
Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:
{d,e,f}, {a,c} and {b,d}

{d,e,f}
- Lower bounds: $\emptyset$, thus no glb
- Upper bounds: $\emptyset$, thus no lub

{a,c}
- Lower bounds: $\emptyset$, thus no glb
- Upper bounds: $\{h\}$, lub: $h$

{b,d}
- Lower bounds: $\{b\}$, glb: $b$
- Upper bounds: $\{d,g\}$, lub: $d$ because $d \preceq g$
Extremal Elements: Example 3

- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b, d, i, j

- Bounds, glb, lub of \{c, e\}?  
  - Lower bounds: \{a, c\}, thus glb is c
  - Upper bounds: \{e, f, g, h, i, j\}, thus lub is e

- Bounds, glb, lub of \{b, i\}?  
  - Lower bounds: \{a\}, thus glb is c
  - Upper bounds: ∅, thus lub DNE
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Lattices

• A special structure arises when every pair of elements in a poset has an lub and a glb

• **Definition:** A lattice is a partially ordered set in which every pair of elements has both
  – a least upper bound and
  – a greatest lower bound
Lattices: Example 1

- Is the example from before a lattice?

- No, because the pair \{b,c\} does not have a least upper bound
Lattices: Example 2

• What if we modified it as shown here?

• Yes, because for any pair, there is an lub & a glb
Lattices: Example 3

- Is this example a lattice?
  
  **No!**
  
  • The lower bound of $A=\{e,f\}$ is $\{a,b,c\}$
  • However, $A$ has no glb

  **Similarly, $B=\{b,c\}$ has no ulb**
A Lattice Or Not a Lattice?

• To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)

• For a pair not to have an lub/glb, the elements of the pair must first be incomparable (Why?)

• You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice
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• **Topological Sorting**
Topological Sorting

• Let us return to the introductory example of Avery Hall renovation. Now that we have got a partial order model, it would be nice to actually create a concrete schedule

• That is, given a partial order, we would like to transform it into a total order that is compatible with the partial order

• A total order is compatible if it does not violate any of the original relations in the partial order

• Essentially, we are simply imposing an order on incomparable elements in the partial order
Topological Sorting: Preliminaries (1)

• Before we give the algorithm, we need some tools to justify its correctness
• **Fact**: Every finite, nonempty poset \((S,\preceq)\) has a minimal element
• We will prove the above fact by a form of *reductio ad absurdum*
Topological Sorting: Preliminaries (2)

**Proof:**

- Assume, to the contrary, that a nonempty finite poset \((S, \prec)\) has no minimal element. In particular, assume that \(a_1\) is not a minimal element.
- Assume, w/o loss of generality, that \(|S| = n\)
- If \(a_1\) is not minimal, then there exists \(a_2\) such that \(a_2 \prec a_1\)
- But \(a_2\) is also not minimal because of the above assumption
- Therefore, there exists \(a_3\) such that \(a_3 \prec a_2\). This process proceeds until we have the last element \(a_n\). Thus, \(a_n \prec a_{n-1} \prec \ldots \prec a_2 \prec a_1\)
- Finally, by definition \(a_n\) is the minimal element \(QED\)
Topological Sorting: Intuition

- The idea of topological sorting is
  - We start with a poset \((S, \preceq)\)
  - We remove a minimal element, choosing arbitrarily if there is more than one. Such an element is guaranteed to exist by the previous fact
  - As we remove each minimal element, one at a time, the set \(S\) shrinks
  - Thus we are guaranteed that the algorithm will **terminate** in a finite number of steps
  - Furthermore, the order in which the elements are removed is a total order: \(a_1 \prec a_2 \prec \ldots \prec a_{n-1} \prec a_n\)

- Now, we can give the algorithm itself
Topological Sorting: Algorithm

*Input*: $(S, \preceq)$ a poset with $|S| = n$

*Output*: A total ordering $(a_1, a_2, \ldots, a_n)$

1. $k \leftarrow 1$
2. **While** $S$ **Do**
3. $a_k \leftarrow$ a minimal element in $S$
4. $S \leftarrow S \setminus \{a_k\}$
5. $k \leftarrow k+1$
6. **End**
7. **Return** $(a_1, a_2, \ldots, a_n)$
Topological Sorting: Example

- Find a compatible ordering (topological ordering) of the poset represented by the Hasse diagrams below.
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