Master Theorem

Section 7.3 of Rosen
Spring 2011
CSCE 235 Introduction to Discrete Structures
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Outline

• Motivation
• The Master Theorem
  – Pitfalls
  – 3 examples
• 4th Condition
  – 1 example
Motivation: Asymptotic Behavior of Recursive Algorithms

• When analyzing algorithms, recall that we only care about the asymptotic behavior
• Recursive algorithms are no different
• Rather than solving exactly the recurrence relation associated with the cost of an algorithm, it is sufficient to give an asymptotic characterization
• The main tool for doing this is the master theorem
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• **The Master Theorem**
  – Pitfalls
  – 3 examples

• 4\textsuperscript{th} Condition
  – 1 example
Master Theorem

Let \( T(n) \) be a monotonically increasing function that satisfies

\[
T(n) = a \cdot T(n/b) + f(n)
\]

\( T(1) = c \)

where \( a \geq 1, \ b \geq 2, \ c > 0 \). If \( f(n) \) is \( \Theta(n^d) \) where \( d \geq 0 \) then

\[
T(n) = \begin{cases} 
\Theta(n^d) & \text{if } a < b^d \\
\Theta(n^d \log n) & \text{if } a = b^d \\
\Theta(n^{\log_b a}) & \text{if } a > b^d 
\end{cases}
\]
Master Theorem: Pitfalls

• You cannot use the Master Theorem if
  – $T(n)$ is not monotone, e.g. $T(n) = \sin(x)$
  – $f(n)$ is not a polynomial, e.g., $T(n)=2T(n/2)+2^n$
  – $b$ cannot be expressed as a constant, e.g.
    \[ T(n) = T(\sqrt{n}) \]

• Note that the Master Theorem does not solve the recurrence equation

• Does the base case remain a concern?
Master Theorem: Example 1

- Let \( T(n) = T(n/2) + \frac{1}{2} n^2 + n \). What are the parameters?
  \[
  a = 1 \\
  b = 2 \\
  d = 2
  \]

  Therefore, which condition applies?
  
  \( 1 < 2^2 \), case 1 applies

- We conclude that
  \[
  T(n) \in \Theta(n^d) = \Theta(n^2)
  \]
Master Theorem: Example 2

- Let $T(n) = 2T(n/4) + \sqrt{n} + 42$. What are the parameters?
  - $a = 2$
  - $b = 4$
  - $d = 1/2$

Therefore, which condition applies?
- $2 = 4^{1/2}$, case 2 applies

- We conclude that

$$T(n) \in \Theta(n^d \log n) = \Theta(\log n \sqrt{n})$$
Master Theorem: Example 3

• Let $T(n) = 3 \cdot T(n/2) + 3/4n + 1$. What are the parameters?
  
  \begin{align*}
  a &= 3 \\
  b &= 2 \\
  d &= 1
  \end{align*}
  
  Therefore, which condition applies?

  $3 > 2^1$, case 3 applies

• We conclude that

$$T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$$

• Note that $\log_2 3 \approx 1.584...$, can we say that $T(n) \in \Theta(n^{1.584})$

  No, because $\log_2 3 \approx 1.5849...$ and $n^{1.584} \notin \Theta(n^{1.5849})$
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‘Fourth’ Condition

- Recall that we cannot use the Master Theorem if \( f(n) \), the non-recursive cost, is not a polynomial.
- There is a limited 4\(^{th}\) condition of the Master Theorem that allows us to consider polylogarithmic functions.
- **Corollary:** If \( f(n) \in \Theta(n^{\log_b a} \log^k n) \) for some \( k \geq 0 \) then \( T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) \).
- This final condition is fairly limited and we present it merely for sake of completeness. Relax ☺️
‘Fourth’ Condition: Example

• Say we have the following recurrence relation
  \[ T(n) = 2 \cdot T(n/2) + n \log n \]

• Clearly, \( a = 2 \), \( b = 2 \), but \( f(n) \) is not a polynomial. However, we have \( f(n) \in \Theta(n \log n) \), \( k = 1 \)

• Therefore by the 4\(^{th} \) condition of the Master Theorem we can say that

\[ T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n^{\log_2 2} \log^2 n) = \Theta(n \log^2 n) \]
Summary

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• The Master Theorem
  – Pitfalls
  – 3 examples
• 4\textsuperscript{th} Condition
  – 1 example