Master Theorem

Section 7.3 of Rosen

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CSCE 235 Introduction to Discrete Structures

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Outline

- Motivation
- The Master Theorem
  - Pitfalls
  - 3 examples
- 4th Condition
  - 1 example

Motivation: Asymptotic Behavior of Recursive Algorithms

- When analyzing algorithms, recall that we only care about the asymptotic behavior
- Recursive algorithms are no different
- Rather than solving exactly the recurrence relation associated with the cost of an algorithm, it is sufficient to give an asymptotic characterization
- The main tool for doing this is the master theorem

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Master Theorem

- Let T(n) be a monotonically increasing function that satisfies

$$T(n) = a\,T(n/b) + f(n)$$
$$T(1) = c$$

where $a \geq 1$, $b \geq 2$, $c>0$. If $f(n)$ is $\Theta(n^d)$ where $d \geq 0$ then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{If } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Master Theorem: Pitfalls

- You cannot use the Master Theorem if
  - T(n) is not monotone, e.g. T(n) = sin(x)
  - f(n) is not a polynomial, e.g., $T(n)=2T(n/2)+2^n$
  - b cannot be expressed as a constant, e.g.

$$T(n) = T(\sqrt{n})$$

- Note that the Master Theorem does not solve the recurrence equation
- Does the base case remain a concern?

Master Theorem: Example 1

- Let $T(n) = T(n/2) + \frac{1}{2} n^2 + n$. What are the parameters?

  a = 1

  b = 2

  d = 2

  Therefore, which condition applies?

  $1 < 2^2$, case 1 applies

- We conclude that

$$T(n) \in \Theta(n^d) = \Theta(n^2)$$

Master Theorem: Example 2

- Let $T(n) = 2\,T(n/4) + \sqrt{n} + 42$. What are the parameters?

  a = 2

  b = 4

  d = 1/2

  Therefore, which condition applies?

  $2 = 4^{1/2}$, case 2 applies

- We conclude that

$$T(n) \in \Theta(n^d \log n) = \Theta(\log n \sqrt{n})$$

Master Theorem: Example 3

- Let T(n)= 3 T(n/2) + 3/4n + 1. What are the parameters?

  a = 3

  b = 2

  d = 1

  Therefore, which condition applies?

  $3 > 2^1$, case 3 applies

- We conclude that

$$T(n) \in \Theta(n^{\log_b a}) = \Theta(n^{\log_2 3})$$

- Note that $\log_2 3 \approx 1.584$..., can we say that $T(n) \in \Theta(n^{1.584})$

  No, because $\log_2 3 \approx 1.5849$... and $n^{1.584} \notin \Theta(n^{1.5849})$

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‘Fourth’ Condition

- Recall that we cannot use the Master Theorem if f(n), the non-recursive cost, is not a polynomial
- There is a limited 4th condition of the Master Theorem that allows us to consider polylogarithmic functions
- **Corollary**: If $f(n) \in \Theta(n^{\log_b a} \log^k n)$ for some k≥0 then

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$$

- This final condition is fairly limited and we present it merely for sake of completeness.. Relax ☺

'Fourth' Condition: Example

- Say we have the following recurrence relation

  T(n)= 2 T(n/2) + n log n

- Clearly, a=2, b=2, but f(n) is not a polynomial. However, we have f(n)∈Θ(n log n), k=1
- Therefore by the 4^{th} condition of the Master Theorem we can say that

$$T(n) \in \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n^{\log_2 2} \log^2 n) = \Theta(n \log^2 n)$$

Summary

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