Introduction to Logic

Sections 1.1 and 1.2 of Rosen

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CSCE 235 Introduction to Discrete Structures

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Introduction: Logic?

- We will study
  - Propositional Logic (PL)
  - First-Order Logic (FOL)
- Logic
  - is the study of the logic <u>relationships</u> between <u>objects</u> and
  - forms the basis of all mathematical reasoning and all automated reasoning

Introduction: PL?

- Propositional Logic (PL) = Propositional Calculus = Sentential Logic
- In Propositional Logic, the objects are called <u>propositions</u>
- **Definition**: A proposition is a <u>statement</u> that is either <u>true</u> or <u>false</u>, but not both
- We usually denote a proposition by a letter: p, q, r, s, …

Outline

- Defining Propositional Logic
  - Propositions
  - Connectives
  - Precedence of Logical Operators
  - Truth tables
- Usefulness of Logic
  - Bitwise operations
  - Logic in Theoretical Computer Science (SAT)
  - Logic in Programming
- Logical Equivalences
  - Terminology
  - Truth tables
  - Equivalence rules

Introduction: Proposition

- **Definition**: The value of a proposition is called its <u>truth value</u>; denoted by
  - *T* or 1 if it is true or
  - *F* or 0 if it is false
- Opinions, interrogative, and imperative are not propositions
- **Truth table**

p
0
1

Propositions: Examples

- The following are propositions
  - Today is Monday *M*
  - The grass is wet *W*
  - It is raining *R*
- The following are not propositions
  - C++ is the best language *Opinion*
  - When is the pretest? *Interrogative*
  - Do your homework *Imperative*

Are these propositions?

- 2+2=5
- Every integer is divisible by 12
- Microsoft is an excellent company

Logical connectives

- Connectives are used to create a compound proposition from two or more propositions
  - Negation (e.g., $\neg a$ or !a or $\bar{a}$) `$\neg$, $\bar$`
  - And or logical conjunction (denoted $\wedge$) `$\wedge$`
  - OR or logical disjunction (denoted $\vee$) `$\vee$`
  - XOR or exclusive or (denoted $\oplus$) `$\oplus$`
  - Impli ion (denoted $\Rightarrow$ or $\rightarrow$) `$\Rightarrow$, $\rightarrow$`
  - Biconditional (denoted $\Leftrightarrow$ or $\leftrightarrow$) `$\LeftRightarrow$, $\leftrightarrow$`
- We define the meaning (semantics) of the logical connectives using <u>truth tables</u>

Precedence of Logical Operators

- As in arithmetic, an ordering is imposed on the use of logical operators in compound propositions
- However, it is preferable to use parentheses to disambiguate operators and facilitate readability

$$\neg\, p \lor q \land \neg\, r \;\equiv\; (\neg p) \lor (q \land (\neg r))$$

- To avoid unnecessary parenthesis, the following precedences hold:
    1. Negation ($\neg$)
    2. Conjunction ($\land$)
    3. Disjunction ($\lor$)
    4. Implication ($\rightarrow$)
    5. Biconditional ($\leftrightarrow$)

Logical Connective: Negation

- $\neg p$, the negation of a proposition p, is also a proposition
- Examples:
  - Today is not Monday
  - It is not the case that today is Monday, etc.
- **Truth table**

p	$\neg p$
0	1
1	0

Logical Connective: Logical And

- The logical connective And is true only when both of the propositions are true. It is also called a <u>conjunction</u>
- Examples
  - It is raining and it is warm
  - (2+3=5) and (1<2)
  - Schroedinger's cat is dead and Schroedinger's cat is not dead.
- Truth table

p	q	$p \wedge q$
0	0	
0	1	
1	0	
1	1	

Logical Connective: Logical OR

- The logical disjunction, or logical OR, is true if one or both of the propositions are true.
- Examples
  - It is raining or it is the second lecture
  - $(2+2=5) \vee (1<2)$
  - You may have cake or ice cream
- **Truth table**

p	***q***	$p \wedge q$	$p \vee q$
0	0	0	
0	1	0	
1	0	0	
1	1	1	

Logical Connective: Exclusive Or

- The exclusive OR, or XOR, of two propositions is true when exactly one of the propositions is true and the other one is false
- Example
  - The circuit is either ON or OFF but not both
  - Let $ab<0$, then either $a<0$ or $b<0$ but not both
  - You may have cake or ice cream, but not both
- Truth table

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$
0	0	0	0	
0	1	0	1	
1	0	0	1	
1	1	1	1	

Logical Connective: Implication (1)

- **Definition:** Let *p* and *q* be two propositions. The implication $p \rightarrow q$ is the proposition that is false when *p* is true and *q* is false and true otherwise
  - *p* is called the hypothesis, antecedent, premise
  - *q* is called the conclusion, consequence
- **Truth table**

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$
0	0	0	0	0	
0	1	0	1	1	
1	0	0	1	1	
1	1	1	1	0	

Logical Connective: Implication (2)

- The implication of $p \rightarrow q$ can be also read as
  - If p then q
  - p implies q
  - If p, q
  - p only if q
  - q if p
  - q when p
  - q whenever p
  - q follows from p
  - p is a sufficient condition for q (p is sufficient for q)
  - q is a necessary condition for p (q is necessary for p)

Logical Connective: Implication (3)

- Examples
  - If you buy you air ticket in advance, it is cheaper.
  - If x is an integer, then $x^2 \geq 0$.
  - If it rains, the grass gets wet.
  - If the sprinklers operate, the grass gets wet.
  - If 2+2=5, then all unicorns are pink.

Exercise: Which of the following implications is true?

- If -1 is a positive number, then 2+2=5

  True. The premise is obviously false, thus no matter what the conclusion is, the implication holds.

- If -1 is a positive number, then 2+2=4

  True. Same as above.

- If *sin* $x = 0$, then $x = 0$

  False. x can be a multiple of π. If we let $x=2\pi$, then *sin* $x=0$ but $x \neq 0$. The implication "if *sin* $x = 0$, then $x = k\pi$, for some k" is true.

Logical Connective: Biconditional (1)

- **Definition:** The biconditional $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values. It is false otherwise.
- Note that it is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$
- **Truth table**

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	0	0	0	1	
0	1	0	1	1	1	
1	0	0	1	1	0	
1	1	1	1	0	1	

Logical Connective: Biconditional (2)

- The biconditional $p \leftrightarrow q$ can be equivalently read as
  - p if and only if q
  - p is a necessary and sufficient condition for q
  - if p then q, and conversely
  - p iff q (Note typo in textbook, page 9, line 3)
- Examples
  - $x>0$ if and only if x^2 is positive
  - The alarm goes off iff a burglar breaks in
  - You may have pudding iff you eat your meat

Exercise: Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if x=0 and y=0

  True. Both implications hold

- 2 + 2 = 4 if and only if $\sqrt{2}<2$

  True. Both implications hold.

- $x^2 \geq 0$ if and only if $x \geq 0$

  False. The implication “if $x \geq 0$ then $x^2 \geq 0$” holds.
  However, the implication “if $x^2 \geq 0$ then $x \geq 0$” is false.
  Consider x=-1.
  The hypothesis $(-1)^2=1 \geq 0$ but the conclusion fails.

Converse, Inverse, Contrapositive

- Consider the proposition $p \rightarrow q$
  - Its <u>converse</u> is the proposition $q \rightarrow p$
  - Its <u>inverse</u> is the proposition $\neg p \rightarrow \neg q$
  - Its <u>contrapositive</u> is the proposition $\neg q \rightarrow \neg p$

Truth Tables

- Truth tables are used to show/define the relationships between the truth values of
  - the individual propositions and
  - the compound propositions based on them

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \Rightarrow q$	$p \Leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Constructing Truth Tables

- Construct the truth table for the following compound proposition

$$((\,p \wedge q\,) \vee \neg q\,)$$

p	q	$p \wedge q$	$\neg q$	$((\,p \wedge q\,) \vee \neg q\,)$
0	0	0	1	1
0	1	0	0	0
1	0	0	1	1
1	1	1	0	1

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  - Precedence of Logical Operators
  - Truth tables
- **Usefulness of Logic**
  - **Bitwise operations**
  - **Logic in Theoretical Computer Science (SAT)**
  - **Logic in Programming**
- Logical Equivalences
  - Terminology
  - Truth tables
  - Equivalence rules

Usefulness of Logic

- Logic is more precise than natural language
  - You may have cake or ice cream.
    - Can I have both?
  - If you buy your air ticket in advance, it is cheaper.
    - Are there or not cheap last-minute tickets?
- For this reason, logic is used for hardware and software <u>specification</u>
  - Given a set of logic statements,
  - One can decide whether or not they are <u>satisfiable</u> (i.e., consistent), although this is a costly process…

Bitwise Operations

- Computers represent information as bits (binary digits)
- A bit string is a sequence of bits
- The length of the string is the number of bits in the string
- Logical connectives can be applied to bit strings of equal length
- Example

	0110 1010 1101
	0101 0010 1111
	_____________
Bitwise OR	0111 1010 1111
Bitwise AND	...
Bitwise XOR	...

Logic in TCS

- **What is SAT?** SAT is the problem of determining whether or not a sentence in propositional logic (PL) is satisfiable.
  - **Given**: a PL sentence
  - **Question**: Determine whether or not it is satisfiable
- Characterizing SAT as an NP-complete problem (complexity class) is at the foundation of Theoretical Computer Science.
- What is a PL sentence? What does satisfiable mean?

Logic in TCS: A Sentence in PL

- A <u>Boolean variable</u> is a variable that can have a value 1 or 0. Thus, Boolean variable is a proposition.
- A <u>term</u> is a Boolean variable
- A <u>literal</u> is a term or its negation
- A <u>clause</u> is a disjunction of literals
- A <u>sentence</u> in PL is a conjunction of clauses
- Example: $(a \vee b \vee \neg c \vee \neg d) \wedge (\neg b \vee c) \wedge (\neg a \vee c \vee d)$
- A sentence in PL is <u>satisfiable</u> iff
  - we can assign a truth value
  - to each Boolean variables
  - such that the sentence evaluates to true (i.e., holds)

SAT in TCS

- Problem
  - **Given**: A sentence in PL (a complex proposition), which is
    - Boolean variables connected with logical connectives
    - Usually, as a conjunction of clauses (CNF = Conjunctive Normal Form)
  - **Question**:
    - Find an assignment of truth values (0/1)
    - That makes the sentence true, i.e. the sentence holds

Logic in Programming: Example 1

- Say you need to define a conditional statement as follows:
  - Increment x if the following condition holds

    $(x > 0 \text{ and } x < 10) \text{ or } x=10$
- You may try: `If (0<x<10 OR x=10) x++;`
- Can't be written in C++ or Java
- How can you modify this statement by using logical equivalence
- Answer: `If (x>0 AND x<=10) x++;`

Logic in Programming: Example 2

- Say we have the following loop

```
While
   ((i<size AND A[i]>10) OR
    (i<size AND A[i]<0) OR
    (i<size AND (NOT (A[i]!=0 AND NOT (A[i]>=10)))))
```

- Is this a good code? Keep in mind:
  - Readability
  - Extraneous code is inefficient and poor style
  - Complicated code is more prone to errors and difficult to debug
  - Solution? Comes later…

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  - Logic in Programming
- **Logical Equivalences**
  - **Terminology**
  - **Truth tables**
  - **Equivalence rules**

Propositional Equivalences: Introduction

- To manipulate a set of statements (here, logical propositions) for the sake of mathematical argumentation, an important step is to replace one statement with another equivalent statement (i.e., with the same truth value)
- Below, we discuss
  – Terminology
  – Establishing logical equivalences using truth tables
  – Establishing logical equivalences using known laws (of logical equivalences)

Terminology: Tautology, Contradictions, Contingencies

- Definitions
  - A compound proposition that is always true, no matter what the truth values of the propositions that occur in it is called a tautology
  - A compound proposition that is always false is called a contradiction
  - A proposition that is neither a tautology nor a contradiction is a contingency
- Examples
  - A simple tautology is $p \vee \neg p$
  - A simple contradiction is $p \wedge \neg p$

Logical Equivalences: Definition

- **Definition**: Propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- Informally, p and q are equivalent if whenever p is true, q is true, and vice versa
- Notation: $p \equiv q$ (p is equivalent to q), $p \leftrightarrow q$, and $p \Leftrightarrow q$
- Alert: $\equiv$ is not a logical connective $\equiv$

Logical Equivalences: Example 1

- Are the propositions ($p \rightarrow q$) and ($\neg p \vee q$) logically equivalent?
- To find out, we construct the truth tables for each:

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
0	0			
0	1			
1	0			
1	1			

The two columns in the truth table are identical, thus we conclude that
$(p \rightarrow q) \equiv (\neg p \vee q)$

Logical Equivalences: Example 1

- Show that (Exercise 25 from Rosen)

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

p	q	r	$p \to r$	$q \to r$	$(p \to r) \lor (q \to r)$	$p \land q$	$(p \land q) \to r$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Propositional Equivalences: Introduction

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- Below, we discuss
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Logical Equivalences: Cheat Sheet

- Table of logical equivalences can be found in Rosen (page 24)
- These and other can be found in a handout on the course web page: http://www.cse.unl.edu/~cse235/files/LogicalEquivalences.pdf
- Let's take a quick look at this Cheat Sheet

Using Logical Equivalences: Example 1

- Logical equivalences can be used to construct additional logical equivalences
- Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

0.	$(p \wedge q) \rightarrow q$	
1.	$\equiv \neg(p \wedge q) \vee q$	Implication Law on 0
2.	$\equiv (\neg p \vee \neg q) \vee q$	De Morgan's Law (1^{st}) on 1
3.	$\equiv \neg p \vee (\neg q \vee q)$	Associative Law on 2
4.	$\equiv \neg p \vee 1$	Negation Law on 3
5.	$\equiv 1$	Domination Law on 4

My Advice

- Remove double implication
- Replace implication by disjunction
- Push negation inwards
- Distribute

Using Logical Equivalences: Example 2

- Example (Exercise 17)*: Show that $\neg(p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$
- Sometimes it helps to start with the second proposition $(p \leftrightarrow \neg q)$

0.	$(p \leftrightarrow \neg q)$	
1.	$\equiv (p \to \neg q) \land (\neg q \to p)$	Equivalence Law on 0
2.	$\equiv (\neg p \lor \neg q) \land (q \lor p)$	Implication Law on 1
3.	$\equiv \neg(\neg((\neg p \lor \neg q) \land (q \lor p)))$	Double negation on 2
4.	$\equiv \neg(\neg(\neg p \lor \neg q) \lor \neg(q \lor p))$	De Morgan's Law…
5.	$\equiv \neg((p \land q) \lor (\neg q \land \neg p))$	De Morgan's Law
6.	$\equiv \neg((p \lor \neg q) \land (p \lor \neg p) \land (q \lor \neg q) \land (q \lor \neg p))$	Distribution Law
7.	$\equiv \neg((p \lor \neg q) \land (q \lor \neg p))$	Identity Law
8.	$\equiv \neg((q \to p) \land (p \to q))$	Implication Law
9.	$\equiv \neg(p \leftrightarrow q)$	Equivalence Law

*See Table 8 (p 25) but you are not allowed to use the table for the proof

Using Logical Equivalences: Example 3

- Show that $\neg(q \rightarrow p) \vee (p \wedge q) \equiv q$

0. $\neg(q \rightarrow p) \vee (p \wedge q)$
1. $\equiv \neg(\neg q \vee p) \vee (p \wedge q)$ — Implication Law
2. $\equiv (q \wedge \neg p) \vee (p \wedge q)$ — De Morgan's & Double negation
3. $\equiv (q \wedge \neg p) \vee (q \wedge p)$ — Commutative Law
4. $\equiv q \wedge (\neg p \vee p)$ — Distributive Law
5. $\equiv q \wedge 1$ — Identity Law

   $\equiv q$ — Identity Law

Proving Logical Equivalences: Summary

- Proving two PL sentences A,B are equivalent using **TT** + **EL**
  1. Verify that the 2 columns of A, B in the truth table are the same (i.e., A,B have the same models)
  2. Verify that the column of ($A \rightarrow B \wedge B \rightarrow A$) in the truth table has all-1 entries (it is a tautology)
  3. Put A,B in CNF, they should be the same
     - Sequence of equivalence laws: Biconditional, implication, moving negation inwards, distributivity
  4. Apply a sequence of inference laws
     - Starting from one sentence, usually the most complex one,
     - Until reaching the second sentence
     - Typical sequence: Biconditional, implication, moving negation inwards, distributivity

Logic in Programming: Example 2 (revisited)

- Recall the loop

```
While
   ((i<size AND A[i]>10) OR
    (i<size AND A[i]<0) OR
    (i<size AND (NOT (A[i]!=0 AND NOT (A[i]>=10)))))
```

- Now, using logical equivalences, simplify it!
- Using De Morgan's Law and Distributivity

```
While ((i<size) AND
         ((A[i]>10 OR A[i]<0) OR
          (A[i]==0 OR A[i]>=10)))
```

- Noticing the ranges of the 4 conditions of `A[i]`

```
While ((i<size) AND (A[i]>=10 OR A[i]<=0))
```

Programming Pitfall Note

- In C, C++ and Java, applying the commutative law is not such a good idea.
- For example, consider accessing an integer array A of size n:

```
if (i<n && A[i]==0) i++;
```

is not equivalent to

```
if (A[i]==0 && i<n) i++;
```