Functions

Section 2.3 of Rosen
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CSCE 235 Introduction to Discrete Structures
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Outline

• Definitions & terminology
  – function, domain, co-domain, image, preimage (antecedent), range, image of a set, strictly increasing, strictly decreasing, monotonic

• Properties
  – One-to-one (injective)
  – Onto (surjective)
  – One-to-one correspondence (bijective)
  – Exercises (5)

• Inverse functions (examples)

• Operators
  – Composition, Equality

• Important functions
  – identity, absolute value, floor, ceiling, factorial
Introduction

• You have already encountered function
  – $f(x,y) = x+y$
  – $f(x) = x$
  – $f(x) = \sin(x)$

• Here we will study functions defined on **discrete domains** and **ranges**

• We may not always be able to write function in a ‘neat way’ as above
Definition: Function

• **Definition**: A function $f$
  
  – from a set $A$ to a set $B$
  
  – is an assignment of exactly one element of $B$ to each element of $A$.

• We write $f(a)=b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a \in A$.

• **Notation**: $f: A \rightarrow B$
  
  which can be read as ‘$f$ maps $A$ to $B$’

• Note the subtlety
  
  – Each and every element of $A$ has a single mapping
  
  – Each element of $B$ may be mapped to by several elements in $A$ or not at all
Terminology

• Let $f: A \rightarrow B$ and $f(a)=b$. Then we use the following terminology:
  – $A$ is the **domain** of $f$, denoted $\text{dom}(f)$
  – $B$ is the **co-domain** of $f$
  – $b$ is the **image** of $a$
  – $a$ is the **preimage** (antecedent) of $b$
  – The **range** of $f$ is the set of all images of elements of $A$, denoted $\text{rng}(f)$
Function: Visualization

**Domain**

**Co-Domain**

**Range**

**Preimage**

A function, $f: A \rightarrow B$

Image, $f(a) = b$
More Definitions (1)

• **Definition:** Let $f_1$ and $f_2$ be two functions from a set $A$ to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 \cdot f_2$ are also function from $A$ to $\mathbb{R}$ defined by:
  
  $-\ (f_1 + f_2)(x) = f_1(x) + f_2(x)$
  $-\ f_1 f_2(x) = f_1(x) f_2(x)$

• **Example:** Let $f_1(x) = x^4 + 2x^2 + 1$ and $f_2(x) = 2 - x^2$
  
  $-\ (f_1 + f_2)(x) = x^4 + 2x^2 + 1 + 2 - x^2 = x^4 + x^2 + 3$
  $-\ f_1 f_2(x) = (x^4 + 2x^2 + 1)(2 - x^2) = -x^6 + 3x^2 + 2$
More Definitions (2)

• **Definition:** Let $f: A \rightarrow B$ and $S \subseteq A$. The image of the set $S$ is the subset of $B$ that consists of all the images of the elements of $S$. We denote the image of $S$ by $f(S)$, so that

$$f(S) = \{ f(s) \mid \forall s \in S \}$$

• Note there that the image of $S$ is a set and not an element.
Image of a set: Example

• Let:
  – \( A = \{a_1, a_2, a_3, a_4, a_5\} \)
  – \( B = \{b_1, b_2, b_3, b_4, b_5\} \)
  – \( f = \{(a_1, b_2), (a_2, b_3), (a_3, b_3), (a_4, b_1), (a_5, b_4)\} \)
  – \( S = \{a_1, a_3\} \)

• Draw a diagram for \( f \)

• What is the:
  – Domain, co-domain, range of \( f \)?
  – Image of \( S \), \( f(S) \)?
More Definitions (3)

• **Definition**: A function \( f \) whose domain and codomain are subsets of the set of real numbers \( \mathbb{R} \) is called
  
  – **strictly increasing** if \( f(x) < f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \).
  
  – **strictly decreasing** if \( f(x) > f(y) \) whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \).

• A function that is increasing or decreasing is said to be **monotonic**
Outline

• Definitions & terminology

• **Properties**
  – One-to-one (injective)
  – Onto (surjective)
  – One-to-one correspondence (bijective)
  – Exercises (5)

• Inverse functions (examples)

• Operators

• Important functions
**Definition: Injection**

- **Definition**: A function $f$ is said to be **one-to-one** or **injective** (or an injection) if
  \[ \forall x \text{ and } y \text{ in in the domain of } f, f(x)=f(y) \implies x=y \]

- Intuitively, an injection simply means that each element in the range has **at most** one preimage (antecedent)

- It is useful to think of the contrapositive of this definition
  \[ x \neq y \implies f(x) \neq f(y) \]
Definition: Surjection

• **Definition**: A function $f: A \rightarrow B$ is called **onto** or **surjective** (or an **surjection**) if

$$\forall b \in B, \exists a \in A \text{ with } f(a) = b$$

• Intuitively, a surjection means that every element in the codomain is mapped into (i.e., it is an image, has an antecedent)

• Thus, the range is the same as the codomain
**Definition: Bijection**

- **Definition**: A function \( f \) is a **one-to-one** correspondence (or a **bijection**), if is both one-to-one (injective) and onto (surjective)
- One-to-one correspondences are important because they endow a function with an **inverse**.
- They also allow us to have a concept cardinality for infinite sets
- Let’s look at a few examples to develop a feel for these definitions...
Functions: Example 1

- Is this a function? Why?
- No, because each of $a_1$, $a_2$ has two images
Functions: Example 2

• Is this a function
  – One-to-one (injective)? Why? No, \( b_1 \) has 2 preimages
  – Onto (surjective)? Why? No, \( b_4 \) has no preimage
Functions: Example 3

• Is this a function
  – One-to-one (injective)? Why? Yes, no $b_i$ has 2 preimages
  – Onto (surjective)? Why? No, $b_4$ has no preimage
Functions: Example 4

- Is this a function
  - One-to-one (injective)? Why? No, \( b_3 \) has 2 preimages
  - Onto (surjective)? Why? Yes, every \( b_i \) has a preimage
Functions: Example 5

• Is this a function
  – One-to-one (injective)?
  – Onto (surjective)?

Thus, it is a bijection or a one-to-one correspondence.
Exercice 1

• Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by
  
  \[ f(x) = 2x - 3 \]

• What is the domain, codomain, range of $f$?
• Is $f$ one-to-one (injective)?
• Is $f$ onto (surjective)?
• Clearly, $\text{dom}(f) = \mathbb{Z}$. To see what the range is, note that:
  
  $b \in \text{rng}(f) \iff b = 2a - 3$, with $a \in \mathbb{Z}$
  
  $\iff b = 2(a-2)+1$
  
  $\iff b$ is odd
Function 1 (cont’d)

- Thus, the range is the set of all odd integers
- Since the range and the codomain are different (i.e., rng($f$) $\neq \mathbb{Z}$), we can conclude that $f$ is not onto (surjective)
- However, $f$ is one-to-one injective. Using simple algebra, we have:

$$ f(x_1) = f(x_2) \implies 2x_1 - 3 = 2x_2 - 3 \implies x_1 = x_2 \quad \text{QED} $$
Exercise 2

• Let $f$ be as before

\[ f(x) = 2x - 3 \]

but now we define $f: \mathbb{N} \rightarrow \mathbb{N}$

• What is the domain and range of $f$?
• Is $f$ onto (surjective)?
• Is $f$ one-to-one (injective)?

• By changing the domain and codomain of $f$, $f$ is not even a function anymore. Indeed, $f(1) = 2 \cdot 1 - 3 = -1 \notin \mathbb{N}$
Exercice 3

• Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by
  
  $f(x) = x^2 - 5x + 5$

• Is this function
  – One-to-one?
  – Onto?
Exercice 3: Answer

- It is not one-to-one (injective)
  \[ f(x_1) = f(x_2) \Rightarrow x_1^2 - 5x_1 + 5 = x_2^2 - 5x_2 + 5 \Rightarrow x_1^2 - 5x_1 = x_2^2 - 5x_2 \]
  \[ \Rightarrow x_1^2 - x_2^2 = 5x_1 - 5x_2 \Rightarrow (x_1 - x_2)(x_1 + x_2) = 5(x_1 - x_2) \]
  \[ \Rightarrow (x_1 + x_2) = 5 \]
  Many \( x_1, x_2 \in \mathbb{Z} \) satisfy this equality. There are thus an infinite number of solutions. In particular, \( f(2) = f(3) = -1 \)

- It is also not onto (surjective).
  The function is a parabola with a global minimum at \((5/2, -5/4)\). Therefore, the function fails to map to any integer less than -1

- What would happen if we changed the domain/codomain?
Exercice 4

• Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by
  $$f(x) = 2x^2 + 7x$$

• Is this function
  – One-to-one (injective)?
  – Onto (surjective)?

• Again, this is a parabola, it cannot be onto (where is the global minimum?)
Exercice 4: Answer

• f(x) is not one-to-one! Indeed:
  \[ f(x_1) = f(x_2) \Rightarrow 2x_1^2 + 7x_1 = 2x_2^2 + 7x_2 \Rightarrow 2x_1^2 - 2x_2^2 = 7x_2 - 7x_1 \]
  \[ \Rightarrow 2(x_1 - x_2)(x_1 + x_2) = 7(x_2 - x_1) \Rightarrow 2(x_1 + x_2) = -7 \Rightarrow (x_1 + x_2) = -7/2 \]

  But \(-7/2 \not\in \mathbb{Z}\). Therefore it must be the case that \(x_1 = x_2\).

  It follows that \(f\) is a one-to-one function. QED

• f(x) is not surjective because \(f(x) = 1\) does not exist

  \[ 2x^2 + 7x = 1 \Rightarrow x(2x + 7) = 1 \text{ the product of two integers is } 1 \text{ if both integers are 1 or -1} \]
  \[ x = 1 \Rightarrow (2x + 7) = 1 \Rightarrow 9 = 1, \text{ impossible} \]
  \[ x = -1 \Rightarrow -1(-2 + 7) = 1 \Rightarrow -5 = 1, \text{ impossible} \]
Exercise 5

• Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by
  $$f(x) = 3x^3 - x$$

• Is this function
  – One-to-one (injective)?
  – Onto (surjective)?
Exercice 5: $f$ is one-to-one

- To check if $f$ is one-to-one, again we suppose that for $x_1, x_2 \in \mathbb{Z}$ we have $f(x_1) = f(x_2)$

$$f(x_1) = f(x_2) \Rightarrow 3x_1^3 - x_1 = 3x_2^3 - x_2$$

$$\Rightarrow 3x_1^3 - 3x_2^3 = x_1 - x_2$$

$$\Rightarrow 3 (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = (x_1 - x_2)$$

$$\Rightarrow (x_1^2 + x_1x_2 + x_2^2) = 1/3$$

which is impossible because $x_1, x_2 \in \mathbb{Z}$

thus, $f$ is one-to-one
Exercice 5: $f$ is not onto

• Consider the counter example $f(a)=1$
• If this were true, we would have
  \[ 3a^3 - a = 1 \Rightarrow a(3a^2 - 1) = 1 \] where $a$ and $(3a^2 - 1) \in \mathbb{Z}$
• The only time we can have the product of two integers equal to 1 is when they are both equal to 1 or -1
• Neither 1 nor -1 satisfy the above equality
  • Thus, we have identified $1 \in \mathbb{Z}$ that does not have an antecedent and $f$ is not onto (surjective)
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• Properties
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  – Composition, Equality

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Inverse Functions (1)

• **Definition**: Let \( f : A \rightarrow B \) be a bijection. The **inverse** function of \( f \) is the function that assigns to an element \( b \in B \) the unique element \( a \in A \) such that \( f(a) = b \)

• The inverse function is denote \( f^{-1} \).

• When \( f \) is a bijection, its inverse exists and

\[
f(a) = b \iff f^{-1}(b) = a
\]
Inverse Functions (2)

• Note that by definition, a function can have an inverse if and only if it is a bijection. Thus, we say that a bijection is invertible.

• Why must a function be bijective to have an inverse?
  – Consider the case where f is not one-to-one (not injective). This means that some element \( b \in B \) has more than one antecedent in \( A \), say \( a_1 \) and \( a_2 \). How can we define an inverse? Does \( f^{-1}(b) = a_1 \) or \( a_2 \)?
  – Consider the case where f is not onto (not surjective). This means that there is some element \( b \in B \) that does not have any preimage \( a \in A \). What is then \( f^{-1}(b) \)?
Inverse Functions: Representation

A function and its inverse

Domain

Co-Domain

A
B

\( f(a) \)

\( f^{-1}(b) \)
Inverse Functions: Example 1

• Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by
  
  $$f(x) = 2x - 3$$

• What is $f^{-1}$?

  1. We must verify that $f$ is invertible, that is, is a bijection. We prove that is one-to-one (injective) and onto (surjective). It is.

  2. To find the inverse, we use the substitution

     • Let $f^{-1}(y) = x$
     • And $y = 2x - 3$, which we solve for $x$. Clearly, $x = (y+3)/2$
     • So, $f^{-1}(y) = (y+3)/2$
Inverse Functions: Example 2

• Let \( f(x) = x^2 \). What is \( f^{-1} \)?
• No domain/codomain has been specified.
• Say \( f: \mathbb{R} \to \mathbb{R} \)
  – Is \( f \) a bijection? Does its inverse exist?
  – Answer: No
• Say we specify that \( f: A \to B \) where
  \[
  A = \{ x \in \mathbb{R} \mid x \leq 0 \} \quad \text{and} \quad B = \{ y \in \mathbb{R} \mid y \geq 0 \}
  \]
  – Is \( f \) a bijection? Does its inverse exist?
  – Answer: Yes, the function becomes a bijection and thus, has an inverse
Inverse Functions: Example 2 (cont’)

• To find the inverse, we let
  – $f^{-1}(y)=x$
  – $y=x^2$, which we solve for $x$

• Solving for $x$, we get $x=\pm\sqrt{y}$, but which one is it?

• Since $\text{dom}(f)$ is all nonpositive and $\text{rng}(f)$ is nonnegative, thus $x$ must be nonpositive and

$$f^{-1}(y)= -\sqrt{y}$$

• From this, we see that the domains/codomains are just as important to a function as the definition of the function itself
Inverse Functions: Example 3

• Let $f(x) = 2^x$
  – What should the domain/codomain be for this function to be a bijection?
  – What is the inverse?
• The function should be $f: \mathbb{R} \rightarrow \mathbb{R}^+$
• Let $f^{-1}(y) = x$ and $y = 2^x$, solving for $x$ we get $x = \log_2(y)$.
  Thus, $f^{-1}(y) = \log_2(y)$
• What happens when we include 0 in the codomain?
• What happens when restrict either sets to $\mathbb{Z}$?
Function Composition (1)

• The value of functions can be used as the input to other functions

• **Definition**: Let \( g: A \rightarrow B \) and \( f: B \rightarrow C \). The **composition** of the functions \( f \) and \( g \) is

\[
(f \circ g)(x) = f(g(x))
\]

• \( f \circ g \) is read as ‘\( f \) circle \( g \)’, or ‘\( f \) composed with \( g \)’, ‘\( f \) following \( g \)’, or just ‘\( f \) of \( g \)’

• In LaTeX: \( \backslash\text{circ} \)
Function Composition (2)

• Because \((f \circ g)(x) = f(g(x))\), the composition \(f \circ g\) cannot be defined unless the range of \(g\) is a subset of the domain of \(f\)

\[ f \circ g \text{ is defined } \iff \text{rng}(g) \subseteq \text{dom}(f) \]

• The order in which you apply a function matters: you go from the inner most to the outer most

• It follows that \(f \circ g\) is in general not the same as \(g \circ f\)
Composition: Graphical Representation

The composition of two functions
Composition: Graphical Representation

The composition of two functions

$\text{(f \circ g)(a)}$

$\text{a}$

$\text{g(a)}$

$\text{f(g(a))}$

A

B

C
Composition: Example 1

• Let $f$, $g$ be two functions on $\mathbb{R} \rightarrow \mathbb{R}$ defined by
  
  $f(x) = 2x - 3$
  
  $g(x) = x^2 + 1$

• What are $f \circ g$ and $g \circ f$?

• We note that
  
  – $f$ is bijective, thus $\text{dom}(f) = \text{rng}(f) = \text{codomain}(f) = \mathbb{R}$
  
  – For $g$, $\text{dom}(g) = \mathbb{R}$ but $\text{rng}(g) = \{x \in \mathbb{R} \mid x \geq 1\} \subseteq \mathbb{R}^+$
  
  – Since $\text{rng}(g) = \{x \in \mathbb{R} \mid x \geq 1\} \subseteq \mathbb{R}^+ \subseteq \text{dom}(f) = \mathbb{R}$, $f \circ g$ is defined
  
  – Since $\text{rng}(f) = \mathbb{R} \subseteq \text{dom}(g) = \mathbb{R}$, $g \circ f$ is defined
Composition: Example 1 (cont’)

• Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$
• $(f \circ g)(x) = f(g(x)) = f(x^2+1) = 2(x^2+1)-3$
  $= 2x^2 - 1$
• $(g \circ f)(x) = g(f(x)) = g(2x-3) = (2x-3)^2 +1$
  $= 4x^2 - 12x + 10$
Function Equality

• Although it is intuitive, we formally define what it means for two functions to be equal

• **Lemma**: Two functions $f$ and $g$ are equal if and only if
  
  – $\text{dom}(f) = \text{dom}(g)$
  
  – $\forall a \in \text{dom}(f) \ (f(a) = g(a))$
Associativity

• The composition of function is not commutative \((f \circ g \neq g \circ f)\), it is associative

• **Lemma**: The composition of functions is an associative operation, that is

\[
(f \circ g) \circ h = f \circ (g \circ h)
\]
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Important Functions: Identity

• **Definition:** The *identity* function on a set $A$ is the function
  \[
iota : A \rightarrow A
  \]
  defined by $\iota(a) = a$ for all $a \in A$.

• One can view the identity function as a composition of a function and its inverse:
  \[
i(\alpha) = (f \circ f^{-1})(\alpha) = (f^{-1} \circ f)(\alpha)
  \]

• Moreover, the composition of any function $f$ with the identity function is itself $f$:
  \[
  (f \circ \iota)(\alpha) = (\iota \circ f)(\alpha) = f(\alpha)
  \]
Inverses and Identity

• The identity function, along with the composition operation, gives us another characterization of inverses when a function has an inverse
• **Theorem**: The functions $f: A \rightarrow B$ and $g: B \rightarrow A$ are inverses if and only if
  
  $$(g \circ f) = \iota_A \text{ and } (f \circ g) = \iota_B$$

  where the $\iota_A$ and $\iota_B$ are the identity functions on sets $A$ and $B$. That is,

  $$\forall a \in A, b \in B \ ( (g(f(a)) = a) \land (f(g(b))) = b \ )$$
Important Functions: Absolute Value

• **Definition:** The **absolute value** function, denoted $|x|$, $f: R \rightarrow \{y \in R \mid y \geq 0\}$. Its value is defined by

$$|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x \leq 0 
\end{cases}$$
Important Functions: Floor & Ceiling

• Definitions:
  – The **floor function**, denoted \([x]\), is a function \(R \rightarrow \mathbb{Z}\). Its values is the **largest integer** that is less than or equal to \(x\)
  – The ceiling function, denoted \([x]\), is a function \(R \rightarrow \mathbb{Z}\). Its values is the **smallest integer** that is greater than or equal to \(x\)

• In LaTeX: \(\lceil x \rceil\), \(\lfloor x \rfloor\)
Important Functions: Floor
Important Functions: Ceiling
Important Function: Factorial

• The factorial function gives us the number of permutations (that is, uniquely ordered arrangements) of a collection of $n$ objects

• **Definition**: The **factorial** function, denoted $n!$, is a function $\mathbb{N} \to \mathbb{N}^+$. Its value is the **product** of the $n$ positive integers

\[ n! = \prod_{i=1}^{i=n} i = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n \]
Factorial Function & Stirling’s Approximation

• The factorial function is defined on a discrete domain
• In many applications, it is useful a continuous version of the function (say if we want to differentiate it)
• To this end, we have the Stirling’s formula

\[ n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]
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