

Predicate Logic and Quantifiers

Sections 1.3 and 1.4 of Rosen

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CSCE 235 Introduction to Discrete Structures

Course web-page: cse.unl.edu/~cse235

Questions: cse235@cse.unl.edu

LaTeX

- Using the package: `\usepackage{amssymb}`
 - Set of natural numbers: \mathbb{N}
 - Set of integer numbers: \mathbb{Z}
 - Set of rational numbers: \mathbb{Q}
 - Set of real numbers: \mathbb{R}
 - Set of complex numbers: \mathbb{C}

Outline

- Introduction
- Terminology:
 - Propositional functions; arguments; arity; universe of discourse
- Quantifiers
 - Definition; using, mixing, negating them
- Logic Programming (Prolog)
- Transcribing English to Logic
- More exercises

Introduction

- Consider the statements:

$$x > 3, x = y + 3, x + y = z$$

- The symbols $>$, $+$, $=$ denote relations between x and 3 , x , y , and 4 , and x, y , and z , respectively
- These relations may hold or not hold depending on the values that x , y , and z may take.
- A **predicate** is a property that is affirmed or denied about the subject (in logic, we say '**variable**' or '**argument**') of a statement
- Consider the statement : 'x is greater than 3'
 - 'x' is the subject
 - 'is greater than 3' is the predicate

Propositional Functions (1)

- To write in Predicate Logic 'x is greater than 3'
 - We introduce a functional symbol for the **predicate** and
 - Put the subject as an **argument** (to the functional symbol):
 $P(x)$
- Terminology
 - $P(x)$ is a statement
 - P is a predicate or propositional function
 - x as an argument
 - $P(\text{Bob})$ is a proposition

Propositional Functions (2)

- Examples:
 - $\text{Father}(x)$: unary predicate
 - $\text{Brother}(x,y)$: binary predicate
 - $\text{Sum}(x,y,z)$: ternary predicate
 - $P(x,y,z,t)$: n-ary predicate

Propositional Functions (3)

- **Definition:** A statement of the form $P(x_1, x_2, \dots, x_n)$ is the value of the propositional symbol P .
- Here: (x_1, x_2, \dots, x_n) is an n -tuple and P is a predicate
- We can think of a propositional function as a function that
 - Evaluates to true or false
 - Takes one or more arguments
 - Expresses a predicate involving the argument(s)
 - Becomes a **proposition** when values are assigned to the arguments

Propositional Functions: Example

- Let $Q(x,y,z)$ denote the statement ' $x^2+y^2=z^2$ '
 - What is the truth value of $Q(3,4,5)$?
 $Q(3,4,5)$ is true
 - What is the truth value of $Q(2,2,3)$?
 $Q(2,3,3)$ is false
 - How many values of (x,y,z) make the predicate true?

There are infinitely many values that make the proposition true, how many right triangles are there?

Universe of Discourse

- Consider the statement ' $x > 3$ ', does it make sense to assign to x the value 'blue'?
- Intuitively, the universe of discourse is the set of all things we wish to talk about; that is the set of all objects that we can sensibly assign to a variable in a propositional function.
- What would be the universe of discourse for the propositional function below be:

EnrolledCSE235(x) = 'x is enrolled in CSE235'

Universe of Discourse: Multivariate functions

- Each variable in an n -tuple (i.e., each argument) may have a different universe of discourse
- Consider an n -ary predicate P :
$$P(r,g,b,c) = \text{'The } rgb\text{-values of the color } c \text{ is } (r,g,b)\text{'}$$
- Example, what is the truth value of
 - $P(255,0,0,red)$
 - $P(0,0,255,green)$
- What are the universes of discourse of (r,g,b,c) ?

Alert

- Propositional Logic (PL)
 - Sentential logic
 - Boolean logic
 - Zero order logic
- First Order Logic (FOL)
 - Predicate logic (PL)

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Quantifiers: Introduction

- The statement ' $x > 3$ ' is not a proposition
- It becomes a proposition
 - When we assign values to the argument: ' $4 > 3$ ' is true, ' $2 < 3$ ' is false, or
 - When we quantify the statement
- Two quantifiers
 - Universal quantifier \forall \forall
the proposition is true for **all** possible values in the universe of discourse
 - Existential quantifier \exists \exists
the proposition is true for **some** value(s) in the universe of discourse

Universal Quantifier: Definition

- **Definition:** The universal quantification of a predicate $P(x)$ is the proposition ' $P(x)$ is true for all values of x in the universe of discourse.' We use the notation: $\forall x P(x)$, which is read 'for all x '.
- If the universe of discourse is finite, say $\{n_1, n_2, \dots, n_k\}$, then the universal quantifier is simply the conjunction of the propositions over all the elements

$$\forall x P(x) \Leftrightarrow P(n_1) \wedge P(n_2) \wedge \dots \wedge P(n_k)$$

Universal Quantifier: Example 1

- Let
 - $P(x)$: 'x must take a discrete mathematics course' and
 - $Q(x)$: 'x is a CS student.'
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statements:
 - “Every CS student must take a discrete mathematics course.”
$$\forall x Q(x) \rightarrow P(x)$$
 - “Everybody must take a discrete mathematics course or be a CS student.”
$$\forall x (P(x) \vee Q(x))$$
 - “Everybody must take a discrete mathematics course and be a CS student.”
$$\forall x (P(x) \wedge Q(x))$$

Are these statements true or false at UNL?

Universal Quantifier: Example 2

- Express in FOL the statement
‘for every x and every y , $x+y>10$ ’
- Answer:
 1. Let $P(x,y)$ be the statement $x+y>10$
 2. Where the universe of discourse for x, y is the set of integers
 3. The statement is: $\forall x \forall y P(x,y)$
- Shorthand: $\forall x,y P(x,y)$

Existential Quantifier: Definition

- **Definition:** The existential quantification of a predicate $P(x)$ is the proposition 'There exists a value x in the universe of discourse such that $P(x)$ is true'
 - Notation: $\exists x P(x)$
 - Reads: 'there exists x '
- If the universe of discourse is finite, say $\{n_1, n_2, \dots, n_k\}$, then the existential quantifier is simply the disjunction of the propositions over all the elements

$$\exists x P(x) \Leftrightarrow P(n_1) \vee P(n_2) \vee \dots \vee P(n_k)$$

Existential Quantifier: Example 1

- Let $P(x,y)$ denote the statement 'x+y=5'
- What does the expression $\exists x \exists y P(x,y)$ mean?
- Which universe(s) of discourse make it true?

Existential Quantifier: Example 2

- Express formally the statement
‘there exists a real solution to $ax^2+bx-c=0$ ’
- Answer:
 1. Let $P(x)$ be the statement $x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$
 2. Where the universe of discourse for x is the set of real numbers.
Note here that a, b, c are fixed constants.
 3. The statement can be expressed as $\exists x P(x)$
- What is the truth value of $\exists x P(x)$, where UoD is \mathbf{R} ?
 - It is false. When $b^2 < 4ac$, there are no real number x that can satisfy the predicate
- What can we do so that $\exists x P(x)$ is true?
 - Change the universe of discourse to the complex numbers, \mathbf{C}

Quantifiers: Truth values

- In general, when are quantified statements true or false?

Statement	True when...	False when...
$\forall x P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x

Mixing quantifiers (1)

- Existential and universal quantifiers can be used together to quantify a propositional predicate. For example:

$$\forall x \exists y P(x,y)$$

is perfectly valid

- Alert:
 - The quantifiers must be read from left to right
 - The order of the quantifiers is important
 - $\forall x \exists y P(x,y)$ is not equivalent to $\exists y \forall x P(x,y)$

Mixing quantifiers (2)

- Consider
 - $\forall x \exists y \text{ Loves}(x,y)$: Everybody loves somebody
 - $\exists y \forall x \text{ Loves}(x,y)$: There is someone loved by everyone
- The two expressions do not mean the same thing
- $(\exists y \forall x \text{ Loves}(x,y)) \rightarrow (\forall x \exists y \text{ Loves}(x,y))$
but the converse does not hold
- However, you can commute similar quantifiers
 - $\forall x \forall y P(x,y)$ is equivalent to $\forall y \forall x P(x,y)$ (thus, $\forall x,y P(x,y)$)
 - $\exists x \exists y P(x,y)$ is equivalent to $\exists y \exists x P(x,y)$ (thus $\exists x,y P(x,y)$)

Mixing Quantifiers: Truth values

Statement	True when...	False when...
$\forall x \forall y P(x,y)$	$P(x,y)$ is true for every pair x,y	There is at least one <i>pair</i> x,y for which $P(x,y)$ is false
$\forall x \exists y P(x,y)$	For every x , there is a y for which $P(x,y)$ is true	There is an x for which $P(x,y)$ is false for every y
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x , there is a y for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Mixing Quantifiers: Example (1)

- Express, in predicate logic, the statement that there is an infinite number of integers
- Answer:
 1. Let $P(x,y)$ be the statement that $x < y$
 2. Let the universe of discourse be the integers, Z
 3. The statement can be expressed by the following

$$\forall x \exists y P(x,y)$$

Mixing Quantifiers: Example (2)

- Express the *commutative law of addition* for R
- We want to express that for every pair of reals, x, y , the following holds: $x+y=y+x$
- Answer:
 1. Let $P(x,y)$ be the statement that $x+y$
 2. Let the universe of discourse be the reals, R
 3. The statement can be expressed by the following

$$\forall x \forall y (P(x,y) \Leftrightarrow P(y,x))$$

Alternatively, $\forall x \forall y (x+y = y+x)$

Mixing Quantifiers: Example (3)

- Express the multiplicative *law* for nonzero reals $R \setminus \{0\}$ (i.e., every nonzero real has an inverse)
- We want to express that for every real number x , there exists a real number y such that $xy=1$
- Answer:

$$\forall x \exists y (xy = 1)$$

Mixing Quantifiers: Example (4)

false mathematical statement

- Does commutativity for subtraction hold over the reals?
- That is: does $x-y=y-x$ for all pairs x,y in R ?
- Express using quantifiers

$$\forall x \forall y (x-y = y-x)$$

Mixing Quantifiers: Example (5)

- Express the statement as a logical expression:
 - “There is a number x such that
 - when it is added to any number, the result is that number and
 - if it is multiplied by any number, the result is x ”
- Answer:
 - Let $P(x,y)$ be the expression “ $x+y=y$ ”
 - Let $Q(x,y)$ be the expression “ $xy=x$ ”
 - The universe of discourse is N, Z, R, Q (but not Z^+)
 - Then the expression is:

$$\exists x \forall y P(x,y) \wedge Q(x,y)$$

Alternatively: $\exists x \forall y (x+y=y) \wedge (xy = x)$

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Binding Variables

- When a quantifier is used on a variable x , we say that x is bound
- If no quantifier is used on a variable in a predicate statement, the variable is called free
- Examples
 - In $\exists x \forall y P(x, y)$, both x and y are bound
 - In $\forall x P(x, y)$, x is bound but y is free
- A statement is called a well-formed formula, when all variables are properly quantified

Binding Variables: Scope

- The set of all variables bound by a common quantifier is called the scope of the quantifier
- For example, in the expression $\exists x, y \forall z P(x, y, z, c)$
 - What is the scope of existential quantifier?
 - What is the scope of universal quantifier?
 - What are the bound variables?
 - What are the free variables?
 - Is the expression a well-formed formula?

Negation

- We can use negation with quantified expressions as we used them with propositions
- **Lemma:** Let $P(x)$ be a predicate. Then the followings hold:

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

- This is essentially the quantified version of De Morgan's Law (when the universe of discourse is finite, this is exactly De Morgan's Law)

Negation: Truth

Truth Values of Negated Quantifiers

Statement	True when...	False when...
$\neg \exists x P(x) \equiv \forall x \neg P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg \forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Negation: Example

- Rewrite the following expression, pushing negation inward:

$$\neg \forall x (\exists y \forall z P(x,y,z) \wedge \exists z \forall y P(x,y,z))$$

- Answer:

$$\exists x (\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x,y,z))$$

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Prolog (1)

- Prolog (Programming in Logic)
 - is a programming language
 - based on (a restricted form of) Predicate Logic (a.k.a. Predicate Calculus and FOL)
- It was developed
 - by the logicians of the Artificial Intelligence community
 - for symbolic reasoning

Prolog (2)

- Prolog allows the users to express facts and rules
- Facts are propositional functions:
 - `student(mia),`
 - `enrolled(mia,cse235),`
 - `instructor(patel,cse235),` etc.
- Rules are implications with conjunctions:
`teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)`
- Prolog answers queries such as:
`?enrolled(mia,cse235)`
`?enrolled(X,cse476)`
`?teaches(X,mia)`
by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5

English into Logic

- Logic is more precise than English
- Transcribing English into Logic and vice versa can be tricky
- When writing statements with quantifiers, usually the correct meaning is conveyed with the following combinations:

Use \forall with \Rightarrow

$\forall x \text{ Lion}(x) \Rightarrow \text{Fierce}(x)$: Every lion is fierce

$\forall x \text{ Lion}(x) \wedge \text{Fierce}(x)$: Everyone is a lion and everyone is fierce

Use \exists with \wedge

$\exists x \text{ Lion}(x) \wedge \text{Vegan}(x)$: Holds when you have at least one vegan lion

$\exists x \text{ Lion}(x) \Rightarrow \text{Vegan}(x)$: Holds when you have vegan people in the universe of discourse (even though there is no vegan lion in the universe of discourse)

More Exercises (1)

- Let $P(x,y)$ denote 'x is a factor of y' where
 - $x \in \{1,2,3,\dots\}$ and $y \in \{2,3,4,\dots\}$
- Let $Q(x,y)$ denote:
 - $\forall x,y [P(x,y) \rightarrow (x=y) \vee (x=1)]$
- Question: When is $Q(x,y)$ true?

Alert...

- Some students wonder if:

$$\forall x,y P(x,y) \equiv (\forall x P(x,y)) \wedge (\forall y P(x,y))$$

- This is certainly not true.
 - In the left-hand side, both x,y are bound.
 - In the right-hand side,
 - In the first predicate, x is bound and y is free
 - In the second predicate, y is bound and x is free
 - Thus, the left-hand side is a proposition, but the right-hand side is not. They cannot be equivalent
- All variables that occur in a propositional function must be bound to turn it into a proposition