

# Combinatorics

**Section 5.1—5.6 7.5—7.6 of Rosen**

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CSCE 235 Introduction to Discrete Structures

Course web-page: [cse.unl.edu/~cse235](http://cse.unl.edu/~cse235)

**Questions:** [cse235@cse.unl.edu](mailto:cse235@cse.unl.edu)

# Motivation

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- Combinatorics is the study of collections of objects. Specifically, counting objects, arrangement, derangement, etc. along with their mathematical properties
- Counting objects is important in order to analyze algorithms and compute discrete probabilities
- Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability
  - A simple example: How many arrangements are there of a deck of 52 cards?
- In addition, combinatorics can be used as a proof technique
  - A combinatorial proof is a proof method that uses counting arguments to prove a statement

# Outline

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- **Introduction**
- **Counting:**
  - **Product rule, sum rule, Principal of Inclusion Exclusion (PIE)**
  - **Application of PIE: Number of onto functions**
- Pigeonhole principle
  - Generalized, probabilistic forms
- Permutations
- Combinations
- Binomial Coefficients
- Generalizations
  - Combinations with repetitions, permutations with indistinguishable objects
- Algorithms
  - Generating combinations (1), permutations (2)
- More Examples

# Product Rule

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- If two events are not mutually exclusive (that is we do them separately), then we apply the product rule
- **Theorem:** Product Rule

Suppose a procedure can be accomplished with two disjoint subtasks. If there are

- $n_1$  ways of doing the first task and
- $n_2$  ways of doing the second task,

then there are  $n_1 \cdot n_2$  ways of doing the overall procedure

# Sum Rule (1)

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- If two events are mutually exclusive, that is, they cannot be done at the same time, then we must apply the sum rule
- **Theorem:** Sum Rule. If
  - an event  $e_1$  can be done in  $n_1$  ways,
  - an event  $e_2$  can be done in  $n_2$  ways, and
  - $e_1$  and  $e_2$  are mutually exclusivethen the number of ways of both events occurring is  $n_1 + n_2$

# Sum Rule (2)

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- There is a natural generalization to any sequence of  $m$  tasks; namely the number of ways  $m$  mutually events can occur

$$n_1 + n_2 + \dots + n_{m-1} + n_m$$

- We can give another formulation in terms of sets. Let  $A_1, A_2, \dots, A_m$  be pairwise disjoint sets. Then

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| \cup |A_2| \cup \dots \cup |A_m|$$

(In fact, this is a special case of the general Principle of Inclusion-Exclusion (PIE))

# Principle of Inclusion-Exclusion (PIE)

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- Say there are two events,  $e_1$  and  $e_2$ , for which there are  $n_1$  and  $n_2$  possible outcomes respectively.
- Now, say that only one event can occur, not both
- In this situation, we cannot apply the sum rule. Why?
  - ... because we would be over counting the number of possible outcomes.
- Instead we have to count the number of possible outcomes of  $e_1$  and  $e_2$  minus the number of possible outcomes in common to both; i.e., the number of ways to do both tasks
- If again we think of them as sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

# PIE (2)

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- More generally, we have the following
- **Lemma:** Let  $A, B$ , be subsets of a finite set  $U$ . Then
  1.  $|A \cup B| = |A| + |B| - |A \cap B|$
  2.  $|A \cap B| \leq \min \{|A|, |B|\}$
  3.  $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
  4.  $|\overline{A}| = |U| - |A|$
  5.  $|A \oplus B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|$
  6.  $|A \times B| = |A| \times |B|$



# PIE: Theorem

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- **Theorem:** Let  $A_1, A_2, \dots, A_n$  be finite sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= \sum_i |A_i| \\ &\quad - \sum_{i < j} |A_i \cap A_j| \\ &\quad + \sum_{i < j < k} |A_i \cap A_j \cap A_k| \\ &\quad - \dots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n| \end{aligned}$$

Each summation is over

- all  $i$ ,
- pairs  $i, j$  with  $i < j$ ,
- triples with  $i < j < k$ , etc.

# PIE Theorem: Example 1

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- To illustrate, when  $n=3$ , we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - [ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| ] \\ &\quad + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

# PIE Theorem: Example 2

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- To illustrate, when  $n=4$ , we have

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| = & |A_1| + |A_2| + |A_3| + |A_4| \\ & - [ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\ & \quad + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4| ] \\ & + [ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\ & \quad + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| ] \\ & - |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

# Application of PIE: Example A (1)

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- How many integers between 1 and 300 (inclusive) are
  - Divisible by at least one of 3,5,7?
  - Divisible by 3 and by 5 but not by 7?
  - Divisible by 5 but by neither 3 or 7?

- Let

$$A = \{n \in \mathbb{Z} \mid (1 \leq n \leq 300) \wedge (3 \mid n)\}$$

$$B = \{n \in \mathbb{Z} \mid (1 \leq n \leq 300) \wedge (5 \mid n)\}$$

$$C = \{n \in \mathbb{Z} \mid (1 \leq n \leq 300) \wedge (7 \mid n)\}$$

- How big are these sets? We use the floor function

$$|A| = \lfloor 300/3 \rfloor = 100$$

$$|B| = \lfloor 300/5 \rfloor = 60$$

$$|C| = \lfloor 300/7 \rfloor = 42$$

# Application of PIE: Example A (2)

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- How many integers between 1 and 300 (inclusive) are divisible by at least one of 3,5,7?

Answer:  $|A \cup B \cup C|$

- By the principle of inclusion-exclusion

$$|A \cup B \cup C| = |A| + |B| + |C| - [ |A \cap B| + |A \cap C| + |B \cap C| ] + |A \cap B \cap C|$$

- How big are these sets? We use the floor function

$$|A| = \lfloor 300/3 \rfloor = 100$$

$$|A \cap B| = \lfloor 300/15 \rfloor = 20$$

$$|B| = \lfloor 300/5 \rfloor = 60$$

$$|A \cap C| = \lfloor 300/21 \rfloor = 14$$

$$|C| = \lfloor 300/7 \rfloor = 42$$

$$|B \cap C| = \lfloor 300/35 \rfloor = 8$$

$$|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$$

- Therefore:

$$|A \cup B \cup C| = 100 + 60 + 42 - (20 + 14 + 8) + 2 = 162$$

# Application of PIE: Example A (3)

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- How many integers between 1 and 300 (inclusive) are divisible by 3 and by 5 but not by 7?

Answer:  $|(A \cap B) \setminus C|$

- By the definition of set-minus

$$|(A \cap B) \setminus C| = |A \cap B| - |A \cap B \cap C| = 20 - 2 = 18$$

- Knowing that

$$|A| = \lfloor 300/3 \rfloor = 100$$

$$|B| = \lfloor 300/5 \rfloor = 60$$

$$|C| = \lfloor 300/7 \rfloor = 42$$

$$|A \cap B| = \lfloor 300/15 \rfloor = 20$$

$$|A \cap C| = \lfloor 300/21 \rfloor = 14$$

$$|B \cap C| = \lfloor 300/35 \rfloor = 8$$

$$|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$$

# Application of PIE: Example A (4)

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- How many integers between 1 and 300 (inclusive) are divisible by 5 but by neither 3 or 7?

Answer:  $|B \setminus (A \cup C)| = |B| - |B \cap (A \cup C)|$

- Distributing B over the intersection

$$\begin{aligned} |B \cap (A \cup C)| &= |(B \cap A) \cup (B \cap C)| \\ &= |B \cap A| + |B \cap C| - |(B \cap A) \cap (B \cap C)| \\ &= |B \cap A| + |B \cap C| - |B \cap A \cap C| \\ &= 20 + 8 - 2 = 26 \end{aligned}$$

- Knowing that

$$|A| = \lfloor 300/3 \rfloor = 100$$

$$|B| = \lfloor 300/5 \rfloor = 60$$

$$|C| = \lfloor 300/7 \rfloor = 42$$

$$|A \cap B| = \lfloor 300/15 \rfloor = 20$$

$$|A \cap C| = \lfloor 300/21 \rfloor = 100$$

$$|B \cap C| = \lfloor 300/35 \rfloor = 8$$

$$|A \cap B \cap C| = \lfloor 300/105 \rfloor = 2$$

# Application of PIE: #Surjections

(Section 7.6)

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- The principle of inclusion-exclusion can be used to count the number of onto (surjective) functions
- **Theorem:** Let  $A, B$  be non-empty sets of cardinality  $m, n$  with  $m \geq n$ . Then there are

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \dots + (-1)^{n-1} \binom{n}{n-1} 1^m$$

*i.e.  $\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m$  onto functions  $f : A \rightarrow B$ .*

$\binom{n}{i}$

*See textbook, Section 7.6 page 509*



# #Surjections: Example

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- How many ways of giving out 6 pieces of candy to 3 children if each child must receive at least one piece?
- This problem can be modeled as follows:
  - Let A be the set of candies,  $|A|=6$
  - Let B be the set of children,  $|B|=3$
  - The problem becomes “find the number of surjective mappings from A to B” (because each child must receive at least one candy)
- Thus the number of ways is thus (m=6, n=3)

$$3^6 - \binom{3}{1}(3-1)^6 + \binom{3}{2}(3-2)^6 = 540$$

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# Pigeonhole Principle (1)

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- If there are more pigeons than there are roots (pigeonholes), for at least one pigeonhole, more than one pigeon must be in it
- **Theorem:** If  $k+1$  or more objects are placed in  $k$  boxes, then there is at least one box containing two or more objects
- This principle is a fundamental tool of elementary discrete mathematics.
- It is also known as the Dirichlet Drawer Principle or Dirichlet Box Principle

# Pigeonhole Principle (2)

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- It is seemingly simple but very powerful
- The difficulty comes in where and how to apply it
- Some simple applications in Computer Science
  - Calculating the probability of hash functions having a collision
  - Proving that there can be no lossless compression algorithm compressing all files to within a certain ration
- **Lemma:** For two finite sets  $A, B$  there exists a bijection  $f:A \rightarrow B$  if and only if  $|A| = |B|$

# Generalized Pigeonhole Principle (1)

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- **Theorem:** If  $N$  objects are placed into  $k$  boxes then there is at least one box containing at least

$$\left\lceil \frac{N}{k} \right\rceil$$

- **Example:** In any group of 367 or more people, at least two of them must have been born on the same date.

# Generalized Pigeonhole Principle (2)

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- A probabilistic generalization states that
  - if  $n$  objects are randomly put into  $m$  boxes
  - with uniform probability
  - (i.e., each object is placed in a given box with probability  $1/m$ )
  - then at least one box will hold more than one object with probability

$$1 - \frac{m!}{(m-n)!m^n}$$

# Generalized Pigeonhole Principle: Example

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- Among 10 people, what is the probability that two or more will have the same birthday?
  - Here  $n=10$  and  $m=365$  (ignoring leap years)
  - Thus, the probability that two will have the same birthday is

$$1 - \frac{365!}{(365 - 10)!365^{10}} \approx 0.1169$$

So, less than 12% probability

# Pigeonhole Principle: Example A (1)

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- Show that
  - in a room of  $n$  people with certain acquaintances,
  - some pair must have the same number of acquaintances
- Note that this is equivalent to showing that any symmetric, irreflexive relation on  $n$  elements must have two elements with the same number of relations
- Proof: by contradiction using the pigeonhole principle
- Assume, to the contrary, that every person has a different number of acquaintances:  $0, 1, 2, \dots, n-1$  (no one can have  $n$  acquaintances because the relation is irreflexive).
- There are  $n$  possibilities, we have  $n$  people, we are not done 😞



# Pigeonhole Principle: Example A (2)

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- Assume, to the contrary, that every person has a different number of acquaintances:  $0, 1, 2, \dots, n-1$  (no one can have  $n$  acquaintances because the relation is irreflexive).
- There are  $n$  possibilities, we have  $n$  people, we are not done 😞
- We need to use the fact that acquaintanceship is a symmetric irreflexive relation
- In particular, some person knows  $0$  people while another knows  $n-1$  people
- This is impossible. Contradiction! 😊 So we do not have  $10$  possibilities, but less
- Thus by the pigeonhole principle ( $10$  people and  $9$  possibilities) at least two people have to the same number of acquaintances

# Pigeonhole Principle: Example B

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- **Example:** Say, 30 buses are to transport 2000 Cornhusker fans to Colorado. Each bus has 80 seats.
- **Show that**
  - One of the buses will have 14 empty seats
  - One of the buses will carry at least 67 passengers
- *One of the buses will have 14 empty seats*
  - Total number of seats is  $80 \cdot 30 = 2400$
  - Total number of empty seats is  $2400 - 2000 = 400$
  - By the pigeonhole principle: 400 empty seats in 30 buses, one must have  $\lceil 400/30 \rceil = 14$  empty seats
- *One of the buses will carry at least 67 passengers*
  - By the pigeonhole principle: 2000 passengers in 30 buses, one must have  $\lceil 2000/30 \rceil = 67$  passengers

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# Permutations

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- A permutation of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of  $r$  elements of a set of  $n$  elements is called an  $r$ -permutation

- **Theorem:** The number of  $r$  permutations of a set of  $n$  distinct elements is

$$P(n, r) = \prod_{i=0}^{r-1} (n - i) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

- It follows that 
$$P(n, r) = \frac{n!}{(n - r)!}$$

- In particular 
$$P(n, n) = n!$$

- Note here that the order is important. It is necessary to distinguish when the order matters and it does not

# Application of PIE and Permutations: Derangements (I)

## (Section 7.6)

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- Consider the hat-check problem
  - Given
    - An employee checks hats from  $n$  customers
    - However, s/he forgets to tag them
    - When customers check out their hats, they are given one at random
  - Question
    - What is the probability that no one will get their hat back?

# Application of PIE and Permutations: Derangements (II)

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- The hat-check problem can be modeled using derangements: permutations of objects such that no element is in its original position
  - Example: 21453 is a derangement of 12345 but 21543 is not
- The number of derangements of a set with  $n$  elements is

$$D_n = n! \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$$

- Thus, the answer to the hatcheck problem is  $\frac{D_n}{n!}$
- Note that 
$$e^{-1} = \left[ 1 - \frac{1}{1!} + \frac{2}{2!} - \frac{3}{3!} + \dots + (-1)^n \frac{1}{n!} \right]$$
- Thus, the probability of the hatcheck problem converges

$$\lim_{n \rightarrow \infty} \frac{D_n}{n!} = e^{-1} \approx 0.368$$

*See textbook, Section 7.6 page 510*

# Permutations: Example A

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- How many pairs of dance partners can be selected from a group of 12 women and 20 men?
  - The first woman can partner with any of the 20 men, the second with any of the remaining 19, etc.
  - To partner all 12 women, we have

$$P(20,12) = 20!/8! = 9.10.11...20$$

# Permutations: Example B

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- In how many ways can the English letters be arranged so that there are exactly 10 letters between a and z?
  - The number of ways is  $P(24,10)$
  - Since we can choose either a or z to come first, then there are  $2P(24,10)$  arrangements of the 12-letter block
  - For the remaining 14 letters, there are  $P(15,15)=15!$  possible arrangements
  - In all there are  $2P(24,10).15!$  arrangements



# Permutations: Example C (1)

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- How many permutations of the letters a, b, c, d, e, f, g contain neither the pattern *bge* nor *eaf*?
  - The total number of permutations is  $P(7,7)=7!$
  - If we fix the pattern *bge*, then we consider it as a single block. Thus, the number of permutations with this pattern is  $P(5,5)=5!$
  - If we fix the pattern *bge*, then we consider it as a single block. Thus, the number of permutations with this pattern is  $P(5,5)=5!$
  - Fixing the patten *eaf*, we have the same number:  $5!$
  - Thus, we have  $(7! - 2 \cdot 5!)$ . Is this correct?
  - No! we have subtracted too many permutations: ones containing both *eaf* and *bfe*.

# Permutations: Example C (2)

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- There are two cases: (1) *eaf* comes first, (2) *bge* comes first
- Are there any cases where *eaf* comes before *bge*?
- No! The letter e cannot be used twice
- If *bge* comes first, then the pattern must be *bgeaf*, so we have 3 blocks or 3! arrangements
- Altogether, we have

$$7! - 2 \cdot (5!) + 3! = 4806$$

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# Combinations (1)

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- Whereas permutations consider order, combinations are used when order does not matter
- **Definition:** A k-combination of elements of a set is an unordered selection of k elements from the set.  
(A combination is imply a subset of cardinality k)

# Combinations (2)

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- **Theorem:** The number of  $k$ -combinations of a set of cardinality  $n$  with  $0 \leq k \leq n$  is

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n - k)!k!}$$

is read 'n choose k'.

$\{n \text{ \choose } k\}$

# Combinations (3)

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- A useful fact about combinations is that they are symmetric

$$\binom{n}{1} = \binom{n}{n-1} \quad \binom{n}{2} = \binom{n}{n-2} \quad \binom{n}{3} = \binom{n}{n-3}$$

- **Corollary:** Let  $n, k$  be nonnegative integers with  $k \leq n$ , then

$$\binom{n}{k} = \binom{n}{n-k}$$

# Combinations: Example A

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- In the Powerball lottery, you pick
  - five numbers between 1 and 55 and
  - A single ‘powerball’ number between 1 and 42How many possible plays are there?
- Here order does not matter
  - The number of ways of choosing 5 numbers is  $\binom{55}{5}$
  - There are 42 possible ways to choose the powerball
  - The two events are not mutually exclusive:  $42 \binom{55}{5}$
  - The odds of winning are  $\frac{1}{42 \binom{55}{5}} < 0.000000006845$

# Combinations: Example B

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- In a sequence of 10 coin tosses, how many ways can 3 heads and 7 tails come up?
  - The number of ways of choosing 3 heads out of 10 coin tosses is  $\binom{10}{3}$
  - It is the same as choosing 7 tails out of 10 coin tosses  $\binom{10}{7} = \binom{10}{3} = 120$
  - ... which illustrates the corollary  $\binom{n}{k} = \binom{n}{n-k}$



# Combinations: Example C

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- How many committees of 5 people can be chosen from 20 men and 12 women
  - If exactly 3 men must be on each committee?
  - If at least 4 women must be on each committee?
- *If exactly three men must be on each committee?*
  - We must choose 3 men and 2 women. The choices are not mutually exclusive, we use the product rule

$$\binom{20}{3} \cdot \binom{12}{2}$$

- *If at least 4 women must be on each committee?*
  - We consider 2 cases: 4 women are chosen and 5 women are chosen. These choices are mutually exclusive, we use the addition rule:

$$\binom{20}{1} \cdot \binom{12}{4} + \binom{20}{0} \cdot \binom{12}{5} = 10,692$$

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# Binomial Coefficients (1)

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- The number of r-combinations  $\binom{n}{r}$  is also called the binomial coefficient
- The binomial coefficients are the coefficients in the expansion of the expression, (multivariate polynomial),

$$(x+y)^n$$

- A binomial is a sum of two terms

# Binomial Coefficients (2)

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- **Theorem: Binomial Theorem**

Let  $x, y$ , be variables and let  $n$  be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

Expanding the summation we have

$$(x + y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Example

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

# Binomial Coefficients: Example

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- What is the coefficient of the term  $x^8y^{12}$  in the expansion of  $(3x+4y)^{20}$ ?

– By the binomial theorem, we have

$$(3x + 4y)^{20} = \sum_{j=0}^{20} \binom{20}{j} (3x)^{n-j} (4y)^j$$

– When  $j=12$ , we have

$$\binom{20}{12} (3x)^8 (4y)^{12}$$

– The coefficient is

$$\binom{20}{12} 3^8 4^{12} = \frac{20!}{12!8!} 3^8 4^{12} = 13866187326750720$$

# Binomial Coefficients (3)

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- Many useful identities and facts come from the Binomial Theorem

- **Corollary:** 
$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0, \quad n \geq 1$$

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n$$

*Equalities are based on  $(1+1)^n=2^n$ ,  $((-1)+1)^n=0^n$ ,  $(1+2)^n=3^n$*

# Binomial Coefficients (4)

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- **Theorem:** Vandermonde's Identity

Let  $m, n, r$  be nonnegative integers with  $r$  not exceeding either  $m$  or  $n$ . Then

$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}$$

- **Corollary:** If  $n$  is a nonnegative integer then  $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$

- **Corollary:** Let  $n, r$  be nonnegative integers,  $r \leq n$ , then

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}$$

# Binomial Coefficients: Pascal's Identity & Triangle

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- The following is known as Pascal's identity which gives a useful identity for efficiently computing binomial coefficients
- **Theorem:** Pascal's Identity

Let  $n, k \in \mathbb{Z}^+$  with  $n \geq k$ , then

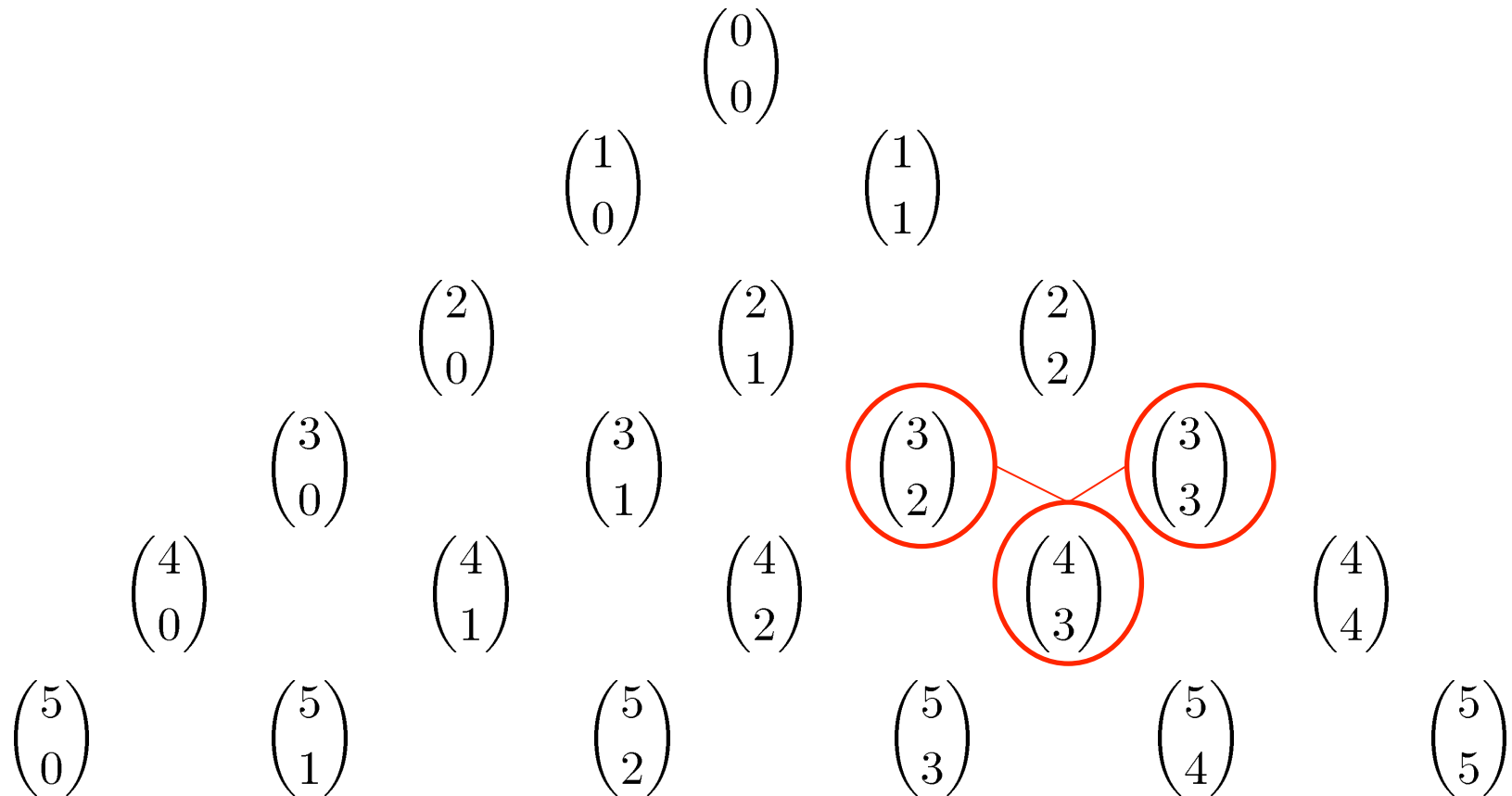
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Identity forms the basis of a geometric object known as Pascal's Triangle



# Pascal's Triangle

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  - Product rule, sum rule, Principal of Inclusion Exclusion (PIE)
  - Application of PIE: Number of onto functions
- Pigeonhole principle
  - Generalized, probabilistic forms
- Permutations
- Combinations
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  - Generating combinations (1), permutations (2)
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# Generalized Combinations & Permutations (1)

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- Sometimes, we are interested in permutations and combinations in which repetitions are allowed
- **Theorem:** The number of  $r$ -permutations of a set of  $n$  objects with repetition allowed is  $n^r$

*...which is easily obtained by the product rule*

- **Theorem:** There are

$$\binom{n + r - 1}{r}$$

$r$ -combinations from a set with  $n$  elements when repetition of elements is allowed

# Generalized Combinations & Permutations: Example

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- There are 30 varieties of donuts from which we wish to buy a dozen. How many possible ways to place your order are there?
- Here,  $n=30$  and we wish to choose  $r=12$ .
- Order does not matter and repetitions are possible
- We apply the previous theorem
- The number of possible orders is

$$\binom{n + r - 1}{r} = \binom{30 + 12 - 1}{12} = \binom{17}{12}$$

## Generalized Combinations & Permutations (2)

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- **Theorem:** The number of different permutations of  $n$  objects where there are  $n_1$  indistinguishable objects of type 1,  $n_2$  of type 2, and  $n_k$  of type  $k$  is

$$\frac{n!}{n_1!n_2! \cdots n_k!}$$

An equivalent ways of interpreting this theorem is the number of ways to

- distribute  $n$  distinguishable objects
- into  $k$  distinguishable boxes
- so that  $n_i$  objects are place into box  $i$  for  $i=1,2,3,\dots,k$

# Example

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- How many permutations of the word Mississippi are there?
- 'Mississippi' has
  - 4 distinct letters: m,i,s,p
  - with 1,4,4,2 occurrences respectively
  - Therefore, the number of permutations is

$$\frac{11!}{1!4!4!2!}$$

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# Algorithms

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- In general, it is inefficient to solve a problem by considering all permutation or combinations since there are exponential (worst, factorial!) numbers of such arrangements
- Nevertheless, for many problems, no better approach is known.
- When exact solutions are needed, backtracking algorithms are used to exhaustively enumerate all arrangements



# Algorithms: Example

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- **Traveling Salesperson Problem (TSP)**

Consider a salesman that must visit  $n$  different cities. He wishes to visit them in an order such that his overall distance travelled is minimized

- This problem is one of hundred of NP-complete problems for which no known efficient algorithms exist. Indeed, it is believed that no efficient algorithms exist. (Actually, Euclidean TSP is not even known to be in NP.)
- The only way of solving this problem exactly is to try all possible  $n!$  routes
- We give several algorithms for generating these combinatorial objects

# Generating Combinations (1)

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- Recall that combinations are simply all possible subsets of size  $r$ . For our purposes, we will consider generating subsets of

$$\{1,2,3,\dots,n\}$$

- The algorithm works as follows
  - Start with  $\{1,\dots,r\}$
  - Assume that we have  $a_1a_2\dots a_r$ , we want the next combination
  - Locate the last element  $a_i$  such that  $a_i \neq n-r-l$
  - Replace  $a_i$  with  $a_i+1$
  - Replace  $a_j$  with  $a_i+j-l$  for  $j=i+1, i+2,\dots,r$

# Generating Combinations (2)

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NEXT R-COMBINATIONS

*Input:* A set of  $n$  elements and an  $r$ -combination  $a_1, a_2, \dots, a_r$

*Output:* The next  $r$ -combination

1.  $i \leftarrow r$
2. **While**  $a_i = n - r + i$  **Do**
3.      $i \leftarrow i - 1$
4. **End**
5.  $a_i \leftarrow a_i + 1$
6. **For**  $j \leftarrow (i + 1)$  **to**  $r$  **Do**
7.      $a_j \leftarrow a_j + j - i$
8. **End**

# Generating Combinations: Example

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- Find the next 3-combination of the set  $\{1,2,3,4,5\}$  after  $\{1,4,5\}$
- Here  $a_1=1$ ,  $a_2=4$ ,  $a_3=5$ ,  $n=5$ ,  $r=3$
- The last  $i$  such that  $a_i \neq 5-3+i$  is 1
- Thus, we set

$$a_1 = a_1 + 1 = 2$$

$$a_2 = a_1 + 2 - 1 = 3$$

$$a_3 = a_1 + 3 - 1 = 4$$

Thus, the next  $r$ -combinations is  $\{2,3,4\}$

# Generating Permutations

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- The textbook gives an algorithm to generate permutations in lexicographic order. Essentially, the algorithm works as follows. Given a permutation
  - Choose the left-most pair  $a_j, a_{j+1}$  where  $a_j < a_{j+1}$
  - Choose the least items to the right of  $a_j$  greater than  $a_j$
  - Swap this item and  $a_j$
  - Arrange the remaining (to the right) items in order

# NEXT PERMUTATION (lexicographic order)

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```
INPUT      : A set of  $n$  elements and an  $r$ -permutation,  $a_1 \cdots a_r$ .
OUTPUT     : The next  $r$ -permutation.

1   $j = n - 1$ 
2  WHILE  $a_j > a_{j+1}$  DO
3       $j = j - 1$ 
4  END
   //  $j$  is the largest subscript with  $a_j < a_{j+1}$ 
5   $k = n$ 
6  WHILE  $a_j > a_k$  DO
7       $k = k - 1$ 
8  END
   //  $a_k$  is the smallest integer greater than  $a_j$  to the right of  $a_j$ 
9   $swap(a_j, a_k)$ 
10  $r = n$ 
11  $s = j + 1$ 
12 WHILE  $r > s$  DO
13      $swap(a_r, a_s)$ 
14      $r = r - 1$ 
15      $s = s + 1$ 
16 END
```

# Generating Permutations (2)

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- Often there is no reason to generate permutations in lexicographic order. Moreover even though generating permutations is inefficient in itself, lexicographic order induces even more work
- An alternate method is to fix an element, then recursively permute the  $n-1$  remaining elements
- The Johnson-Trotter algorithm has the following attractive properties. Not in your textbook, not on the exam, just for your reference/culture
  - It is bottom up (non-recursive)
  - It induces a minimal-change between each permutation

# Johnson-Trotter Algorithm

---

- We associate a direction to each element, for example

$$\overrightarrow{3} \overleftarrow{2} \overrightarrow{4} \overleftarrow{1}$$

- A component is mobile if its direction points to an adjacent component that is smaller than itself.
- Here 3 and 4 are mobile, 1 and 2 are not



# Algorithm: Johnson Trotter

---

INPUT : *An integer  $n$ .*

OUTPUT : *All possible permutations of  $\langle 1, 2, \dots, n \rangle$ .*

**1**  $\pi = \overleftarrow{1} \overleftarrow{2} \dots \overleftarrow{n}$

**2** WHILE *There exists a mobile integer  $k \in \pi$*  DO

**3**      $k =$  *largest mobile integer*

**4**     *swap  $k$  and the adjacent integer  $k$  points to*

**5**     *reverse direction of all integers  $> k$*

**6**     *Output  $\pi$*

**7** END

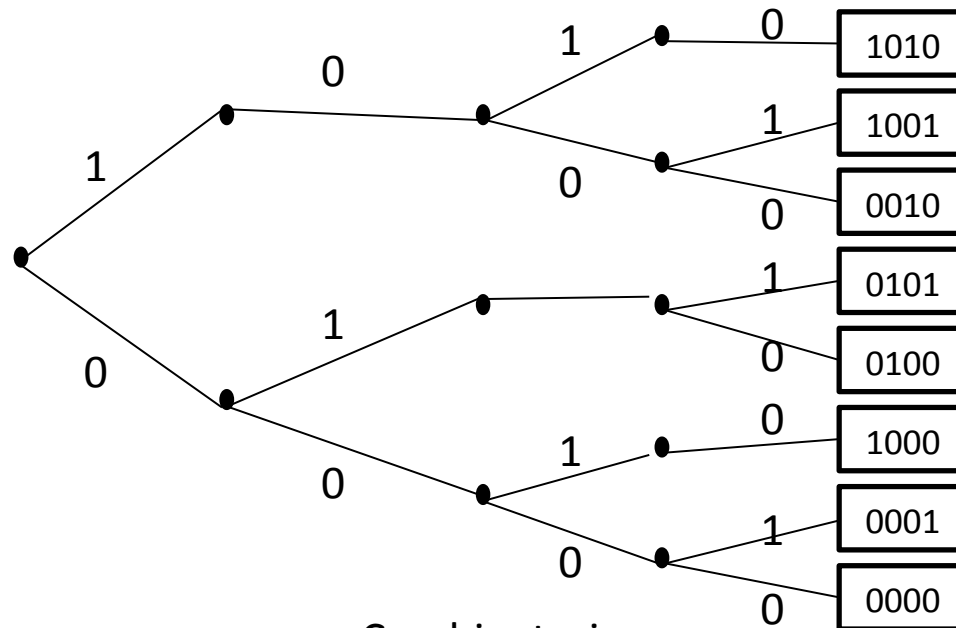
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# Example A

- How many bit strings of length 4 are there such that 11 never appear as a substring
- We can represent the set of strings graphically using a diagram tree (see textbook pages 343)



# Example: Counting Functions (1)

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- Let  $S, T$  be sets such that  $|S|=n$ ,  $|T|=m$ .
  - How many function are there mapping  $f:S \rightarrow T$ ?
  - How many of these functions are one-to-one (injective)?
- A function simply maps each  $s_i$  to one  $t_j$ , thus for each  $n$  we can choose to send it to any of the elements in  $T$
- Each of these is an independent event, so we apply the multiplication rule:
- If we wish  $f$  to be injective, we must have  $n \leq m$ , otherwise the answer is obviously 0

# Example: Counting Functions (2)

---

- Now each  $s_i$  must be mapped to a unique element in  $T$ .
  - For  $s_1$ , we have  $m$  choices
  - However, once we have made a mapping, say  $s_j$ , we cannot map subsequent elements to  $t_j$  again
  - In particular, for the second element,  $s_2$ , we now have  $m-1$  choices, for  $s_3$ ,  $m-2$  choices, etc.

$$m \cdot (m - 1) \cdot (m - 2) \cdot \dots \cdot (m - (n - 2)) \cdot (m - (n - 1))$$

- An alternative way of thinking is using the choose operator: we need to choose  $n$  element from a set of size  $m$  for our mapping

$$\binom{m}{n} = \frac{m!}{(m - n)!n!}$$

- Once we have chosen this set, we now consider all permutations of the mapping, that is  $n!$  different mappings for this set. Thus, the number of such mapping is

$$\frac{m!}{(m - n)!n!} \cdot n! = \frac{m!}{(m - n)!}$$

# Another Example: Counting Functions

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- Let  $S=\{1,2,3\}$ ,  $T=\{a,b\}$ .
  - How many onto (surjective) mappings are there from  $S\rightarrow T$ ?
  - How many onto-to-one injective functions are there from  $T\rightarrow S$ ?
- See Theorem 1, page 509

# Example: Sets

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- How many  $k$  integers  $1 \leq k \leq 100$  are divisible by 2 or 3?
- Let
  - $A = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \wedge (2 \mid n)\}$
  - $B = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \wedge (3 \mid n)\}$
- Clearly,  $|A| = \lfloor 100/2 \rfloor = 50$ ,  $|B| = \lfloor 100/3 \rfloor = 33$
- Do we have  $|A \cup B| = 83$ ? No!
- We have over counted the integers divisible by 6
  - Let  $C = \{n \in \mathbb{Z} \mid (1 \leq n \leq 100) \wedge (6 \mid n)\}$ ,  $|C| = \lfloor 100/6 \rfloor = 16$
- So  $|A \cup B| = (50 + 33) - 16 = 67$

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