Possible Solutions of Homework 7 Proof Problems

Problem: 1.7.22
Conjecture: If $a \neq b$ and $a, b \in \mathbb{R}^+$, the quadratic mean is always greater than their arithmetic mean.
Proof: We want to verify that $\sqrt{\frac{a^2+b^2}{2}} > \frac{a+b}{2}$
That can be re-written by squaring both sides and multiplying by 4:
$2a^2 + 2b^2 > (a+b)^2$
After doing some algebra, this can be written as:
$(a-b)^2 > 0$.
This is always true for $a \neq b$ and $a, b \in \mathbb{R}^+$. So, our conjecture was valid.

Problem: 1.7.28
If $|y| \geq 2$ then, $cx^2 + 5y^2 \geq 2x^2 + 20 \geq 20$. So, the only possible values of y are 0, ± 1. In the former case we look for solutions to $2x^2 = 14$ and in the latter, solutions to $2x^2 = 9$. For both of these, there are no integer solutions.

Problem: 1.7.30
Following the hint, we let, $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$. Then, $x^2+y^2 = (m^2-n^2)^2+(2mn)^2 = m^4-2m^2n^2+n^4+4m^2n^2 = m^4+2m^2n^2+n^4 = (m^2+n^2)^2 = z^2$. Thus, this last equality $x^2 + y^2 = z^2$ would hold for any values of m, n which implies that we have infinitely many solutions to the problem.

Problem: 1.7.34
The *average* of any two numbers is between both of those two numbers. That means average of $a, b \in \mathbb{R}$ and $a < b$ is $\frac{a+b}{2}$ where $a < \frac{a+b}{2} < b$. So, the

average of a rational and an irrational number would be (1) between those two numbers and (2) an irrational number. So, between every rational and an irrational number, there is an irrational number.

Problem: 1.7.32 (Supplemental Exercise)

Proof by contraposition: We need to show that if x is rational x^3 is rational.
If x is rational, we can write $x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.
Then, $x^3 = \frac{a^3}{b^3}$ which is a rational number.

Problem: 1.7.34 (Supplemental Exercise)

We claim that 7 is not a sum of *at most* two squares and a cube. The first two squares are 1 and 4 and the first positive cube is 1. 7 cannot be written as the sum of these three numbers.