Sets

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Spring 2006

Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 1.6 – 1.7 of Rosen

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Introduction I

We've already implicitly dealt with sets (integers, $\mathbb{Z}$; rationals ($\mathbb{Q}$) etc.) but here we will develop more fully the definitions, properties and operations of sets.

Definition

A *set* is an unordered collection of (unique) objects.

Sets are fundamental discrete structures that form the basis of more complex discrete structures like graphs.

Contrast this definition with the one in the book (compare *bag*, *multi-set*, *tuples*, etc).

Introduction II

Definition

The objects in a set are called *elements* or *members* of a set. A set is said to *contain* its elements.

Recall the notation: for a set A, an element x we write

$$x \in A$$

if A contains x and

$$x \notin A$$

otherwise.

Latex notation: \in, \neg\in.

Terminology I

Definition

Two sets, A and B are *equal* if they contain the same elements. In this case we write $A = B$.

Example

$\{2, 3, 5, 7\} = \{3, 2, 7, 5\}$ since a set is *unordered*.
Also, $\{2, 3, 5, 7\} = \{2, 2, 3, 3, 5, 7\}$ since a set contains *unique* elements.
However, $\{2, 3, 5, 7\} \neq \{2, 3\}$.

Terminology II

A multi-set is a set where you specify the number of occurrences of each element: $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ is a set where m_1 occurs a_1 times, m_2 occurs a_2 times, etc.

Note in CS (Databases), we distinguish:

- a set is w/o repetition
- a bag is a set with repetition

Terminology III

We've already seen *set builder* notation:

$$O = \{x \mid (x \in \mathbb{Z}) \wedge (x = 2k \text{ for some } k \in \mathbb{Z})\}$$

should be read O is the set that contains all x such that x is an integer and x is even.

A set is defined in *intension*, when you give its set builder notation.

$$O = \{x \mid (x \in \mathbb{Z}) \wedge (x \leq 8)\}$$

A set is defined in *extension*, when you enumerate all the elements.

$$O = \{0, 2, 6, 8\}$$

A set can also be represented graphically using a *Venn diagram*.

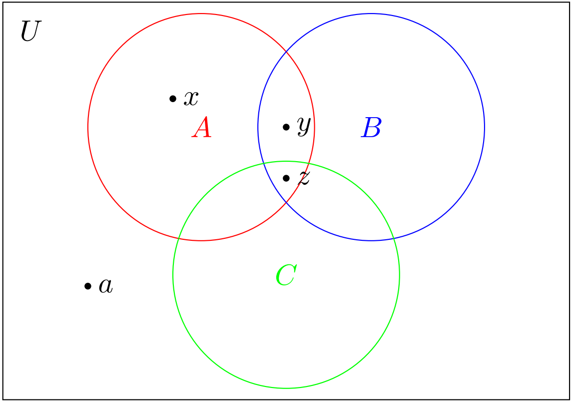


Figure: Venn Diagram

More Terminology & Notation I

A set that has no elements is referred to as the *empty set* or *null set* and is denoted $\emptyset$.

A *singleton* set is a set that has only one element. We usually write $\{a\}$. Note the different: brackets indicate that the object is a *set* while a without brackets is an *element*.

The subtle difference also exists with the empty set: that is

$$\emptyset \neq \{\emptyset\}$$

The first is a set, the second is a set containing a set.

More Terminology & Notation II

Definition

A is said to be a subset of B and we write

$$A \subseteq B$$

if and only if every element of A is also an element of B.

That is, we have an equivalence:

$$A \subseteq B \iff \forall x(x \in A \rightarrow x \in B)$$

Theorem

For any set S,

- $\emptyset \subseteq S$ *and*
- $S \subseteq S$

(Theorem 1, page 81.)

The proof is in the book—note that it is an excellent example of a vacuous proof!

Latex notation: \emptyset, \subset, \subseteq.

More Terminology & Notation IV

Definition

A set A that is a subset of B is called a *proper subset* if $A \neq B$. That is, there is some element $x \in B$ such that $x \notin A$. In this case we write $A \subset B$ or to be even more definite we write

$$A \subsetneq B$$

Example

Let $A = \{2\}$. Let $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$. Then $A \subsetneq B$.

Latex notation: \subsetneq.

More Terminology & Notation V

Sets can be elements of other sets.

Example

$$\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

and

$$\{\{1\}, \{2\}, \{3\}\}$$

are sets with sets for elements.

More Terminology & Notation VI

Definition

If there are exactly n distinct elements in a set S, with n a nonnegative integer, we say that S is a finite set and the cardinality of S is n. Notationally, we write

$$|S| = n$$

Definition

A set that is not finite is said to be *infinite*.

More Terminology & Notation VII

Example

Recall the set $B = \{x \mid (x \leq 100) \wedge (x \text{ is prime})\}$, its cardinality is

$$|B| = 25$$

since there are 25 primes less than 100. Note the cardinality of the empty set:

$$|\emptyset| = 0$$

The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all infinite.

Proving Equivalence I

You may be asked to show that a set is a subset, proper subset or equal to another set. To do this, use the equivalence discussed before:

$$A \subseteq B \iff \forall x(x \in A \to x \in B)$$

To show that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B. Any proof method could be used.

To show that $A \subsetneq B$ you must show that A is a subset of B just as before. But you must also show that

$$\exists x((x \in B) \wedge (x \notin A))$$

Proving Equivalence II

Finally, to show two sets equal, it is enough to show (much like an equivalence) that $A \subseteq B$ and $B \subseteq A$ independently.

Logically speaking this is showing the following quantified statements:

$$\big(\forall x (x \in a \rightarrow x \in B)\big) \wedge \big(\forall x (x \in B \rightarrow x \in A)\big)$$

We'll see an example later.

The Power Set I

Definition

The *power set* of a set S, denoted $\mathcal{P}(S)$ is the set of all subsets of S.

Example

Let $A = \{a, b, c\}$ then the power set is

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Note that the empty set and the set itself are always elements of the power set. This follows from Theorem 1 (Rosen, p81).

The Power Set II

The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set.

Fact

Let S be a set such that $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n$$

Tuples I

Sometimes we may need to consider ordered collections.

Definition

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection with the a_i being the i-th element for $i = 1, 2, \ldots, n$.

Two ordered n-tuples are equal if and only if for each $i = 1, 2, \ldots, n$, $a_i = b_i$.

For $n = 2$, we have *ordered pairs*.

Cartesian Products I

Definition

Let A and B be sets. The *Cartesian product* of A and B denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$$

The Cartesian product is also known as the *cross product*.

Definition

A subset of a Cartesian product, $R \subseteq A \times B$ is called a *relation*. We will talk more about relations in the next set of slides.

Note that $A \times B \neq B \times A$ unless $A = \emptyset$ or $B = \emptyset$ or $A = B$. Can you find a counter example to prove this?

Sets

CSE235

Cartesian products can be generalized for any n-tuple.

Definition

The *Cartesian product* of n sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Notation With Quantifiers

Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language.

Now we can simplify things using set notation!

Example

$$\forall x \in \mathbb{R}(x^2 \geq 0)$$

$$\exists x \in \mathbb{Z}(x^2 = 1)$$

Or you can mix quantifiers:

$$\forall a, b, c \in \mathbb{R}\, \exists x \in \mathbb{C}(ax^2 + bx + c = 0)$$

Set Operations

Just as arithmetic operators can be used on pairs of numbers, there are operators that can act on sets to give us new sets.

Definition

The *union* of two sets A and B is the set that contains all elements in A, B or both. We write

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

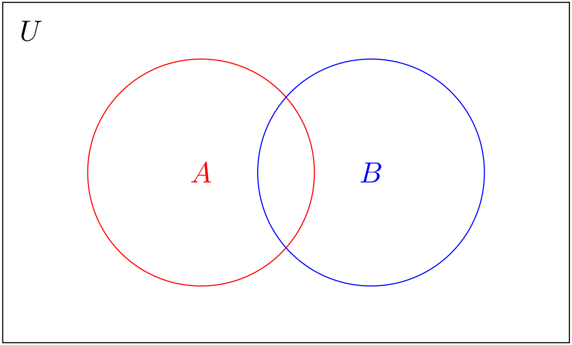
Latex notation: \cup.

Definition

The *intersection* of two sets A and B is the set that contains all elements that are elements of *both* A *and* B We write

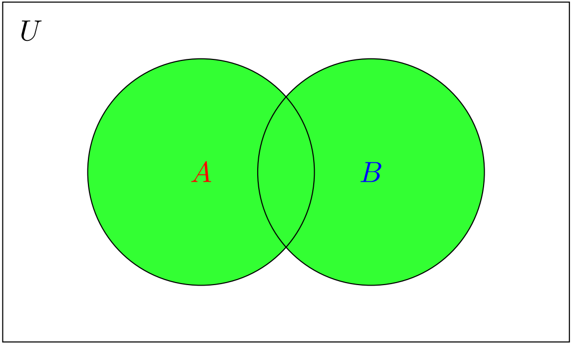
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Latex notation: \cap.



Sets A and B

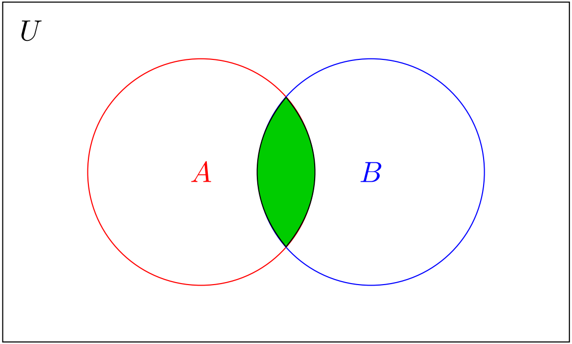
Set Operators
Venn Diagram Example: Union



Union, $A \cup B$

Set Operators

Venn Diagram Example: Intersection



Intersection, $A \cap B$

Disjoint Sets

Definition

Two sets are said to be *disjoint* if their intersection is the empty set: $A \cap B = \emptyset$

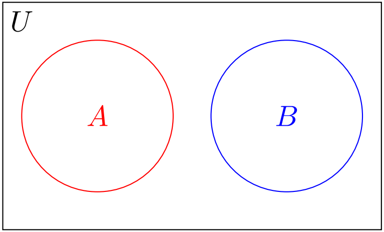


Figure: Two disjoint sets A and B.

Set Difference

Definition

The *difference* of sets A and B, denoted by $A \setminus B$ (or $A - B$) is the set containing those elements that are in A but not in B.

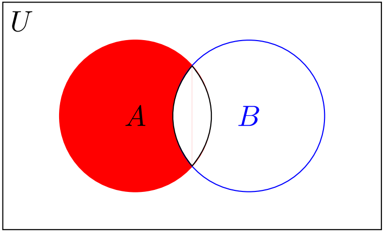


Figure: Set Difference, $A \setminus B$

Latex notation: \setminus.

Set Complement

Definition

The *complement* of a set A, denoted $\bar{A}$, consists of all elements *not* in A. That is, the difference of the universal set and A; $U \setminus A$.

$$\bar{A} = \{x \mid x \notin A\}$$

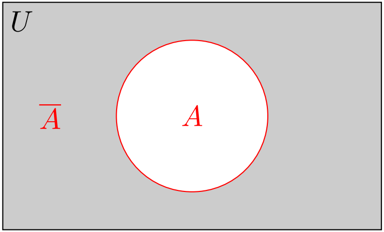


Figure: Set Complement, $\overline{A}$

Set Identities

There are analogs of all the usual laws for set operations. Again, the Cheat Sheet is available on the course web page.

http://www.cse.unl.edu/cse235/files/LogicalEquivalences.pdf

Let's take a quick look at this Cheat Sheet

Recall that to *prove* such an identity, one must show that

1. The left hand side is a subset of the right hand side.
2. The right hand side is a subset of the left hand side.
3. Then conclude that they are, in fact, equal.

The book proves several of the standard set identities. We'll give a couple of different examples here.

Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of } 3\}$ and $C = \{x \mid x \text{ is a multiple of } 6\}$. Show that

$$A \cap B = C$$

Proof.

$\square$

Proving Set Equivalences

Example I

Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of } 3\}$ and $C = \{x \mid x \text{ is a multiple of } 6\}$. Show that

$$A \cap B = C$$

Proof.

$(A \cap B \subseteq C)$: Let $x \in A \cap B$. Then x is a multiple of 2 *and* x is a multiple of 3, therefore we can write $x = 2 \cdot 3 \cdot k$ for some integer k. Thus $x = 6k$ and so x is a multiple of 6 and $x \in C$.

□

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$(C \subseteq A \cap B)$: Let $x \in C$. Then x is a multiple of 6 and so $x = 6k$ for some integer k. Therefore $x = 2(3k) = 3(2k)$ and so $x \in A$ and $x \in B$. It follows then that $x \in A \cap B$ by definition of intersection, thus $C \subseteq A \cap B$. $\square$

Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of } 3\}$ and $C = \{x \mid x \text{ is a multiple of } 6\}$. Show that

$$A \cap B = C$$

Proof.

$(A \cap B \subseteq C)$: Let $x \in A \cap B$. Then x is a multiple of 2 *and* x is a multiple of 3, therefore we can write $x = 2 \cdot 3 \cdot k$ for some integer k. Thus $x = 6k$ and so x is a multiple of 6 and $x \in C$.
$(C \subseteq A \cap B)$: Let $x \in C$. Then x is a multiple of 6 and so $x = 6k$ for some integer k. Therefore $x = 2(3k) = 3(2k)$ and so $x \in A$ and $x \in B$. It follows then that $x \in A \cap B$ by definition of intersection, thus $C \subseteq A \cap B$.
We conclude that $A \cap B = C$ $\square$

An alternative prove uses *membership tables* where an entry is 1 if it a chosen (but fixed) element is in the set and 0 otherwise.

Example

(Exercise 13, p95): Show that

$$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$$

Proving Set Equivalences

Example II Continued

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
0	0	0						
0	0	1						
0	1	0						
0	1	1						
1	0	0						
1	0	1						
1	1	0						
1	1	1						

1 under a set indicates that an element is in the set.

If the columns are equivalent, we can conclude that indeed,

$$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$$

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
0	0	0	0					
0	0	1	0					
0	1	0	0					
0	1	1	0					
1	0	0	0					
1	0	1	0					
1	1	0	0					
1	1	1	1					

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Proving Set Equivalences

Example II Continued

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
0	0	0	0	1				
0	0	1	0	1				
0	1	0	0	1				
0	1	1	0	1				
1	0	0	0	1				
1	0	1	0	1				
1	1	0	0	1				
1	1	1	1	0				

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Proving Set Equivalences

Example II Continued

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0	0	0	0	1	1			
0	0	1	0	1	1			
0	1	0	0	1	1			
0	1	1	0	1	1			
1	0	0	0	1	0			
1	0	1	0	1	0			
1	1	0	0	1	0			
1	1	1	1	0	0			

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Proving Set Equivalences
Example II Continued

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0	0	0	0	1	1	1		
0	0	1	0	1	1	1		
0	1	0	0	1	1	0		
0	1	1	0	1	1	0		
1	0	0	0	1	0	1		
1	0	1	0	1	0	1		
1	1	0	0	1	0	0		
1	1	1	1	0	0	0		

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Proving Set Equivalences

Example II Continued

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0	0	0	0	1	1	1	1	
0	0	1	0	1	1	1	0	
0	1	0	0	1	1	0	1	
0	1	1	0	1	1	0	0	
1	0	0	0	1	0	1	1	
1	0	1	0	1	0	1	0	
1	1	0	0	1	0	0	1	
1	1	1	1	0	0	0	0	

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0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	0	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0

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Proving Set Equivalences

Example II Continued

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\bar{A}$	$\bar{B}$	$\bar{C}$	$\bar{A} \cup \bar{B} \cup \bar{C}$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	1	1	0	1
0	1	0	0	1	1	0	1	1
0	1	1	0	1	1	0	0	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	0	1	0	1
1	1	0	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0

1 under a set indicates that an element is in the set.

Since the columns are equivalent, we conclude that indeed,

$$\overline{A \cap B \cap C} = \bar{A} \cup \bar{B} \cup \bar{C}$$

Generalized Unions & Intersections I

In the previous example we showed that De Morgan's Law generalized to unions involving 3 sets. Indeed, for any finite number of sets, De Morgan's Laws hold.

Moreover, we can generalize set operations in a straightforward manner to any finite number of sets.

Definition

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Latex notation: \bigcup.

Generalized Unions & Intersections II

Definition

The *intersection* of a collection of sets is the set that contains those elements that are members of *every* set in the collection.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Latex notation: \bigcap.

Computer Representation of Sets I

There really aren't ways to represent *infinite* sets by a computer since a computer is has a finite amount of memory (unless of course, there is a finite *representation*).

If we assume that the universal set U is finite, however, then we can easily and efficiently represent sets by *bit vectors*.

Specifically, we force an ordering on the objects, say

$$U = \{a_1, a_2, \ldots, a_n\}$$

For a set $A \subseteq U$, a bit vector can be defined as

$$b_i = \begin{cases} 0 & \text{if } a_i \notin A \\ 1 & \text{if } a_i \in A \end{cases}$$

for $i = 1, 2, \ldots, n$.

Computer Representation of Sets II

Example

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $A = \{0, 1, 6, 7\}$ Then the bit vector representing A is

$$1100\ 0011$$

What's the empty set? What's U?

Set operations become almost trivial when sets are represented by bit vectors.

In particular, the bit-wise $\textsc{Or}$ corresponds to the union operation. The bit-wise $\textsc{And}$ corresponds to the intersection operation.

Computer Representation of Sets III

Example

Let U and A be as before and let $B = \{0, 4, 5\}$ Note that the bit vector for B is 1000 1100. The union, $A \cup B$ can be computed by

$$1100\ 0011 \vee 1000\ 1100 = 1100\ 1111$$

The intersection, $A \cap B$ can be computed by

$$1100\ 0011 \wedge 1000\ 1100 = 1000\ 0000$$

What sets do these represent?

Note: If you want to represent *arbitrarily* sized sets, you can still do it with a computer—how?

Conclusion

Questions?