We’ve already implicitly dealt with sets (integers, $\mathbb{Z}$; rationals ($\mathbb{Q}$) etc.) but here we will develop more fully the definitions, properties and operations of sets.

**Definition**

A set is an unordered collection of (unique) objects.

Sets are fundamental discrete structures that form the basis of more complex discrete structures like graphs.

Contrast this definition with the one in the book (compare *bag*, *multi-set*, *tuples*, etc).
Definition

The objects in a set are called *elements* or *members* of a set. A set is said to *contain* its elements.

Recall the notation: for a set \( A \), an element \( x \) we write

\[
x \in A
\]

if \( A \) contains \( x \) and

\[
x \notin A
\]

otherwise.

Latex notation: \( \text{\LaTeX} \text{ in}, \text{ \neg in} \).
**Definition**
Two sets, $A$ and $B$ are *equal* if they contain the same elements. In this case we write $A = B$.

**Example**
\{2, 3, 5, 7\} = \{3, 2, 7, 5\} since a set is *unordered*.
Also, \{2, 3, 5, 7\} = \{2, 2, 3, 3, 5, 7\} since a set contains *unique* elements.
However, \{2, 3, 5, 7\} \neq \{2, 3\}.
A multi-set is a set where you specify the number of occurrences of each element: \( \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\} \) is a set where \( m_1 \) occurs \( a_1 \) times, \( m_2 \) occurs \( a_2 \) times, etc.

Note in CS (Databases), we distinguish:

- a set is w/o repetition
- a bag is a set with repetition
We’ve already seen set builder notation:

\[ O = \{ x \mid (x \in \mathbb{Z}) \land (x = 2k \text{ for some } k \in \mathbb{Z}) \} \]

should be read \( O \) is the set that contains all \( x \) such that \( x \) is an integer and \( x \) is even.

A set is defined in **intension**, when you give its set builder notation.

\[ O = \{ x \mid (x \in \mathbb{Z}) \land (x \leq 8) \} \]

A set is defined in **extension**, when you enumerate all the elements.

\[ O = \{0, 2, 6, 8\} \]
A set can also be represented graphically using a Venn diagram.

Figure: Venn Diagram
A set that has no elements is referred to as the *empty set* or *null set* and is denoted $\emptyset$.

A *singleton* set is a set that has only one element. We usually write $\{a\}$. Note the different: brackets indicate that the object is a *set* while $a$ without brackets is an *element*.

The subtle difference also exists with the empty set: that is

$$\emptyset \neq \{\emptyset\}$$

The first is a set, the second is a set containing a set.
A is said to be a subset of \( B \) and we write

\[
A \subseteq B
\]

if and only if every element of \( A \) is also an element of \( B \).

That is, we have an equivalence:

\[
A \subseteq B \iff \forall x(x \in A \rightarrow x \in B)
\]
For any set $S$,
- $\emptyset \subseteq S$ and
- $S \subseteq S$

(Theorem 1, page 81.)

The proof is in the book—note that it is an excellent example of a vacuous proof!

Latex notation: $\emptyset, \subseteq, \subseteqq$. 
### Definition

A set $A$ that is a subset of $B$ is called a *proper subset* if $A \neq B$. That is, there is some element $x \in B$ such that $x \notin A$. In this case we write $A \subset B$ or to be even more definite we write $A \subsetneq B$.

\[
A \subsetneq B
\]

### Example

Let $A = \{2\}$. Let $B = \{x \mid (x \leq 100) \land (x \text{ is prime})\}$. Then $A \subsetneq B$.

Latex notation: \textbackslash\subsetneq.
Sets can be elements of other sets.

Example

\[
\{\emptyset, \{a\}, \{b\}, \{a, b\}\}
\]

and

\[
\{\{1\}, \{2\}, \{3\}\}
\]

are sets with sets for elements.
Definition
If there are exactly \( n \) distinct elements in a set \( S \), with \( n \) a nonnegative integer, we say that \( S \) is a finite set and the cardinality of \( S \) is \( n \). Notationally, we write

\[
|S| = n
\]

Definition
A set that is not finite is said to be \textit{infinite}. 
Example

Recall the set \( B = \{ x \mid (x \leq 100) \land (x \text{ is prime}) \} \), its cardinality is

\[ |B| = 25 \]

since there are 25 primes less than 100. Note the cardinality of the empty set:

\[ |\emptyset| = 0 \]

The sets \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) are all infinite.
You may be asked to show that a set is a subset, proper subset or equal to another set. To do this, use the equivalence discussed before:

\[ A \subseteq B \iff \forall x (x \in A \to x \in B) \]

To show that \( A \subseteq B \) it is enough to show that for an arbitrary (nonspecific) element \( x, x \in A \) implies that \( x \) is also in \( B \). Any proof method could be used.

To show that \( A \subsetneq B \) you must show that \( A \) is a subset of \( B \) just as before. But you must also show that

\[ \exists x ((x \in B) \land (x \notin A)) \]
Finally, to show two sets equal, it is enough to show (much like an equivalence) that $A \subseteq B$ and $B \subseteq A$ independently.

Logically speaking this is showing the following quantified statements:

$$(\forall x(x \in a \rightarrow x \in B)) \land (\forall x(x \in B \rightarrow x \in A))$$

We’ll see an example later.
The Power Set

**Definition**

The *power set* of a set $S$, denoted $\mathcal{P}(S)$ is the set of all subsets of $S$.

**Example**

Let $A = \{a, b, c\}$ then the power set is

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Note that the empty set and the set itself are always elements of the power set. This follows from Theorem 1 (Rosen, p81).
The power set is a fundamental combinatorial object useful when considering all possible combinations of elements of a set.

**Fact**

Let $S$ be a set such that $|S| = n$, then

$$|\mathcal{P}(S)| = 2^n$$
Sometimes we may need to consider ordered collections.

**Definition**

The *ordered* \( n \)-*tuple* \( (a_1, a_2, \ldots, a_n) \) is the ordered collection with the \( a_i \) being the \( i \)-th element for \( i = 1, 2, \ldots, n \).

Two ordered \( n \)-tuples are equal if and only if for each \( i = 1, 2, \ldots, n \), \( a_i = b_i \).

For \( n = 2 \), we have *ordered pairs*. 
### Definition

Let \( A \) and \( B \) be sets. The *Cartesian product* of \( A \) and \( B \) denoted \( A \times B \), is the set of all ordered pairs \((a, b)\) where \( a \in A \) and \( b \in B \):

\[
A \times B = \{ (a, b) \mid (a \in A) \land (b \in B) \}
\]

The Cartesian product is also known as the *cross product*.

### Definition

A subset of a Cartesian product, \( R \subseteq A \times B \) is called a *relation*. We will talk more about relations in the next set of slides.

Note that \( A \times B \neq B \times A \) unless \( A = \emptyset \) or \( B = \emptyset \) or \( A = B \). Can you find a counter example to prove this?
Cartesian products can be generalized for any $n$-tuple.

**Definition**

The *Cartesian product* of $n$ sets, $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$ is

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$
Whenever we wrote $\exists x P(x)$ or $\forall x P(x)$, we specified the universe of discourse using explicit English language.

Now we can simplify things using set notation!

Example

$\forall x \in \mathbb{R} (x^2 \geq 0)$

$\exists x \in \mathbb{Z} (x^2 = 1)$

Or you can mix quantifiers:

$\forall a, b, c \in \mathbb{R} \exists x \in \mathbb{C} (ax^2 + bx + c = 0)$
Just as arithmetic operators can be used on pairs of numbers, there are operators that can act on sets to give us new sets.
Set Operators
Union

Definition
The union of two sets $A$ and $B$ is the set that contains all elements in $A$, $B$ or both. We write

$$A \cup B = \{ x \mid (x \in A) \lor (x \in B) \}$$

Latex notation: \cup.
Set Operators

Intersection

Definition

The *intersection* of two sets $A$ and $B$ is the set that contains all elements that are elements of *both* $A$ and $B$. We write

$$A \cap B = \{x \mid (x \in A) \land (x \in B)\}$$

Latex notation: \textbackslash cap.
Set Operators
Venn Diagram Example

Sets $A$ and $B$
Set Operators
Venn Diagram Example: Union

Union, \( A \cup B \)
Set Operators

Venn Diagram Example: Intersection

Intersection, $A \cap B$
Disjoint Sets

**Definition**

Two sets are said to be *disjoint* if their intersection is the empty set: \( A \cap B = \emptyset \)

**Figure:** Two disjoint sets \( A \) and \( B \).
Set Difference

**Definition**

The *difference* of sets $A$ and $B$, denoted by $A \setminus B$ (or $A - B$) is the set containing those elements that are in $A$ but not in $B$.

**Figure:** Set Difference, $A \setminus B$

**Latex notation:** \setminus.
Set Complement

**Definition**

The *complement* of a set $A$, denoted $\bar{A}$, consists of all elements *not in* $A$. That is, the difference of the universal set and $A$; $U \setminus A$.

$$\bar{A} = \{ x \mid x \notin A \}$$

**Figure:** Set Complement, $\bar{A}$
Set Identities

There are analogs of all the usual laws for set operations. Again, the Cheat Sheet is available on the course web page.

http://www.cse.unl.edu/cse235/files/LogicalEquivalences.pdf

Let’s take a quick look at this Cheat Sheet
Proving Set Equivalences

Recall that to prove such an identity, one must show that

1. The left hand side is a subset of the right hand side.
2. The right hand side is a subset of the left hand side.
3. Then conclude that they are, in fact, equal.

The book proves several of the standard set identities. We’ll give a couple of different examples here.
Let $A = \{x \mid x \text{ is even}\}$ and $B = \{x \mid x \text{ is a multiple of 3}\}$ and $C = \{x \mid x \text{ is a multiple of 6}\}$. Show that

$$A \cap B = C$$

Proof.

$(A \cap B \subseteq C)$: Let $x \in A \cap B$. Then $x$ is a multiple of 2 and $x$ is a multiple of 3, therefore we can write $x = 2 \cdot 3 \cdot k$ for some integer $k$. Thus $x = 6k$ and so $x$ is a multiple of 6 and $x \in C$.

$(C \subseteq A \cap B)$: Let $x \in C$. Then $x$ is a multiple of 6 and so $x = 6k$ for some integer $k$. Therefore $x = 2(3k) = 3(2k)$ and so $x \in A$ and $x \in B$. It follows then that $x \in A \cap B$ by definition of intersection, thus $C \subseteq A \cap B$.

We conclude that $A \cap B = C$. 
Let \( A = \{ x \mid x \text{ is even} \} \) and \( B = \{ x \mid x \text{ is a multiple of 3} \} \) and \( C = \{ x \mid x \text{ is a multiple of 6} \} \). Show that

\[
A \cap B = C
\]

**Proof.**

\((A \cap B \subseteq C)\): Let \( x \in A \cap B \). Then \( x \) is a multiple of 2 and \( x \) is a multiple of 3, therefore we can write \( x = 2 \cdot 3 \cdot k \) for some integer \( k \). Thus \( x = 6k \) and so \( x \) is a multiple of 6 and \( x \in C \).
Let \( A = \{ x \mid x \text{ is even} \} \) and \( B = \{ x \mid x \text{ is a multiple of } 3 \} \) and \( C = \{ x \mid x \text{ is a multiple of } 6 \} \). Show that 

\[
A \cap B = C
\]

**Proof.**

\((A \cap B \subseteq C)\): Let \( x \in A \cap B \). Then \( x \) is a multiple of 2 and \( x \) is a multiple of 3, therefore we can write \( x = 2 \cdot 3 \cdot k \) for some integer \( k \). Thus \( x = 6k \) and so \( x \) is a multiple of 6 and \( x \in C \).

\((C \subseteq A \cap B)\): Let \( x \in C \). Then \( x \) is a multiple of 6 and so \( x = 6k \) for some integer \( k \). Therefore \( x = 2(3k) = 3(2k) \) and so \( x \in A \) and \( x \in B \). It follows then that \( x \in A \cap B \) by definition of intersection, thus \( C \subseteq A \cap B \).
Proving Set Equivalences

Example I

Let \( A = \{ x \mid x \text{ is even} \} \) and \( B = \{ x \mid x \text{ is a multiple of 3} \} \) and \( C = \{ x \mid x \text{ is a multiple of 6} \} \). Show that

\[
A \cap B = C
\]

Proof.

\((A \cap B \subseteq C)\): Let \( x \in A \cap B \). Then \( x \) is a multiple of 2 and \( x \) is a multiple of 3, therefore we can write \( x = 2 \cdot 3 \cdot k \) for some integer \( k \). Thus \( x = 6k \) and so \( x \) is a multiple of 6 and \( x \in C \).

\((C \subseteq A \cap B)\): Let \( x \in C \). Then \( x \) is a multiple of 6 and so \( x = 6k \) for some integer \( k \). Therefore \( x = 2(3k) = 3(2k) \) and so \( x \in A \) and \( x \in B \). It follows then that \( x \in A \cap B \) by definition of intersection, thus \( C \subseteq A \cap B \).

We conclude that \( A \cap B = C \).
An alternative prove uses *membership tables* where an entry is 1 if it a chosen (but fixed) element is in the set and 0 otherwise.

**Example**

(Exercise 13, p95): Show that

\[
A \cap B \cap C = \bar{A} \cup \bar{B} \cup \bar{C}
\]
### Proving Set Equivalences

**Example II Continued**

We have the following table:

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1 under a set indicates that an element is in the set.

*If* the columns are equivalent, we can conclude that indeed,

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Proving Set Equivalences
Example II Continued

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1 under a set indicates that an element is in the set.

If the columns are equivalent, we can conclude that indeed,

$$A \cap B \cap C = \overline{A} \cup \overline{B} \cup \overline{C}$$
Proving Set Equivalences
Example II Continued

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Proving Set Equivalences
Example II Continued

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In the previous example we showed that De Morgan’s Law generalized to unions involving 3 sets. Indeed, for any finite number of sets, De Morgan’s Laws hold.

Moreover, we can generalize set operations in a straightforward manner to any finite number of sets.

Definition

The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

Latex notation: \( \bigcup \).
Generalized Unions & Intersections II

Definition

The intersection of a collection of sets is the set that contains those elements that are members of every set in the collection.

\[ \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n \]

Latex notation: \( \bigcap \).
There really aren’t ways to represent infinite sets by a computer since a computer has a finite amount of memory (unless of course, there is a finite representation).

If we assume that the universal set $U$ is finite, however, then we can easily and efficiently represent sets by bit vectors.

Specifically, we force an ordering on the objects, say

$$U = \{a_1, a_2, \ldots, a_n\}$$

For a set $A \subseteq U$, a bit vector can be defined as

$$b_i = \begin{cases} 
0 & \text{if } a_i \not\in A \\
1 & \text{if } a_i \in A 
\end{cases}$$

for $i = 1, 2, \ldots, n$. 
Computer Representation of Sets II

Example

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $A = \{0, 1, 6, 7\}$ Then the bit vector representing $A$ is

$$1100\ 0011$$

What’s the empty set? What’s $U$?

Set operations become almost trivial when sets are represented by bit vectors.

In particular, the bit-wise OR corresponds to the union operation. The bit-wise AND corresponds to the intersection operation.
Example

Let $U$ and $A$ be as before and let $B = \{0, 4, 5\}$ Note that the bit vector for $B$ is 1000 1100. The union, $A \cup B$ can be computed by

$$1100\ 0011\ \lor\ 1000\ 1100 = 1100\ 1111$$

The intersection, $A \cap B$ can be computed by

$$1100\ 0011\ \land\ 1000\ 1100 = 1000\ 0000$$

What sets do these represent?

Note: If you want to represent arbitrarily sized sets, you can still do it with a computer—how?
Conclusion

Questions?