Sequences & Summations

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Though you should be (at least intuitively) familiar with sequences and summations, we give a quick review.

Definition

A *sequence* is a function from a subset of integers to a set S. We use the notation(s):

$$\{a_n\} \quad \{a_n\}_n^{\infty} \quad \{a_n\}_{n=0}^{\infty} \quad \{a_n\}_{n=0}^{\infty}$$

Each a_n is called the n-th *term* of the sequence.

We rely on context to distinguish between a sequence and a set; though there is a connection.

Example

Consider the sequence

$$\left\{ \left(1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty}$$

The terms are

$$\begin{aligned} a_1 &= (1+1)^1 &= 2.00000 \\ a_2 &= (1+\tfrac{1}{2})^2 &= 2.25000 \\ a_3 &= (1+\tfrac{1}{3})^3 &= 2.37037 \\ a_4 &= (1+\tfrac{1}{4})^4 &= 2.44140 \\ a_5 &= (1+\tfrac{1}{5})^5 &= 2.48832 \end{aligned}$$

What is this sequence?

The sequence corresponds to e:

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^n\right\} = e = 2.71828\ldots$$

Example

The sequence

$$\{h_n\}_{n=1}^{\infty} = \frac{1}{n}$$

is known as the *harmonic sequence*.

The sequence is simply

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

This sequence is particularly interesting because its summation is *divergent*;

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Definition

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$$

Where $a \in \mathbb{R}$ is called the *initial term* and $r \in \mathbb{R}$ is the *common ratio*.

A geometric progression is a *discrete* analogue of the exponential function

$$f(x) = ar^x$$

Progressions II

Definition

An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, a+3d, \ldots, a+nd, \ldots$$

Where $a \in \mathbb{R}$ is called the *initial term* and $r \in \mathbb{R}$ is the *common difference*.

Again, an arithmetic progression is a discrete analogue of the linear function,

$$f(x) = dx + a$$

Progressions III

Example

A common geometric progression in computer science is

$$\{a_n\} = \frac{1}{2^n}$$

Here, $a = 1$ and $r = \frac{1}{2}$

Table 1 on Page 228 (Rosen) has useful sequences.

Summations I

You should be very familiar with Summation notation:

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_{n-1} + a_n$$

Here, j is the *index of summation*, m is the *lower limit*, and n is the *upper limit*.

Often times, it is useful to change the lower/upper limits; which can be done in a straightforward manner (though we must be careful).

$$\sum_{j=1}^{n} a_j = \sum_{j=0}^{n-1} a_{j+1}$$

Sometimes we can express a summation in *closed form*. Geometric series, for example:

Theorem

For $a, r \in \mathbb{R}, r \neq 0$,

$$\sum_{i=0}^{n} ar^i = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Summations III

Double summations often arise when analyzing an algorithm.

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{i} a_j \quad = \quad & a_1 + \\ & a_1 + a_2 + \\ & a_1 + a_2 + a_3 + \\ & \ldots \\ & a_1 + a_2 + a_3 + \cdots + a_n\end{aligned}$$

Summations can also be indexed over *elements in a set*.

$$\sum_{s \in S} f(s)$$

Table 2 on Page 232 (Rosen) has useful summations.

Series

When we take the sum of a sequence, we get a *series*. We've already seen a closed form for geometric series.

Some other useful closed forms include the following.

$$\begin{aligned}
\sum_{i=l}^{u} 1 &= u - l + 1, \text{ for } l \leq u \\
\sum_{i=0}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=0}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=0}^{n} i^k &\approx \frac{1}{k+1} n^{k+1}
\end{aligned}$$

Infinite Series I

Though we will mostly deal with finite series (i.e. an upper limit of n for a fixed integer), *infinite series* are also useful.

Example

Consider the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

This series converges to 2. However, the geometric series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots$$

does not converge. However, note that $\sum_{n=0}^{n} 2^n = 2^{n+1} - 1$

Infinite Series II

In fact, we can generalize this as follows.

Lemma

A geometric series converges if and only if the absolute value of the common ratio is less than 1.