Sequences & Summations

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Sequences
Definition
A sequence is a function from a subset of integers to a set \( S \). We use the notation(s):
\[
\{ a_n \} \quad \{ a_n \}_{n=0}^{\infty} \quad \{ a_n \}_{n=0}^{\infty}
\]
Each \( a_n \) is called the \( n \)-th term of the sequence.

We rely on context to distinguish between a sequence and a set; though there is a connection.

Sequences
Example
Consider the sequence
\[
\left\{ \left( 1 + \frac{1}{n} \right)^n \right\}_{n=1}^{\infty}
\]
The terms are
\[
a_1 = (1 + 1)^1 = 2.00000
a_2 = (1 + \frac{1}{2})^2 = 2.25000
a_3 = (1 + \frac{1}{3})^3 = 2.37037
a_4 = (1 + \frac{1}{4})^4 = 2.44140
a_5 = (1 + \frac{1}{5})^5 = 2.48832
\]
What is this sequence?

Sequences
Example II
The sequence
\[
\{ h_n \}_{n=1}^{\infty} = \frac{1}{n}
\]
is known as the harmonic sequence.

The sequence is simply
\[
\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \cdots
\]
This sequence is particularly interesting because its summation is divergent;
\[
\sum_{n=1}^{\infty} \frac{1}{n} = \infty
\]
**Progressions I**

**Definition**

A **geometric progression** is a sequence of the form

\[ a, ar, ar^2, ar^3, \ldots, ar^n, \ldots \]

Where \( a \in \mathbb{R} \) is called the *initial term* and \( r \in \mathbb{R} \) is the *common ratio*.

A geometric progression is a discrete analogue of the exponential function

\[ f(x) = ar^x \]

**Progressions II**

**Definition**

An **arithmetic progression** is a sequence of the form

\[ a, a + d, a + 2d, a + 3d, \ldots, a + nd, \ldots \]

Where \( a \in \mathbb{R} \) is called the *initial term* and \( r \in \mathbb{R} \) is the *common difference*.

Again, an arithmetic progression is a discrete analogue of the linear function,

\[ f(x) = dx + a \]

**Progressions III**

**Example**

A common geometric progression in computer science is

\[ \{a_n\} = \frac{1}{2^n} \]

Here, \( a = 1 \) and \( r = \frac{1}{2} \)

Table 1 on Page 228 (Rosen) has useful sequences.

**Summations I**

You should be very familiar with Summation notation:

\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \]

Here, \( j \) is the *index of summation*, \( m \) is the *lower limit*, and \( n \) is the *upper limit*.

Often times, it is useful to change the lower/upper limits; which can be done in a straightforward manner (though we must be careful).

\[ \sum_{j=1}^{n} a_j = \sum_{j=0}^{n-1} a_{j+1} \]

Sometimes we can express a summation in *closed form*. Geometric series, for example:

**Theorem**

For \( a, r \in \mathbb{R} \), \( r \neq 0 \),

\[ \sum_{i=0}^{n} ar^i = \begin{cases} a \frac{r^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases} \]

**Summations II**

Double summations often arise when analyzing an algorithm.

\[ \sum_{i=1}^{n} \sum_{j=1}^{i} a_j = a_1 + a_1 + a_2 + a_1 + a_2 + a_3 + \cdots + a_1 + a_2 + a_3 + \cdots + a_n \]

Summations can also be indexed over *elements in a set*.

\[ \sum_{s \in S} f(s) \]

Table 2 on Page 232 (Rosen) has useful summations.
Series

When we take the sum of a sequence, we get a series. We’ve already seen a closed form for geometric series. Some other useful closed forms include the following.

\[
\sum_{i=l}^{u} 1 = u - l + 1, \text{ for } l \leq u
\]
\[
\sum_{i=0}^{n} i = \frac{n(n + 1)}{2}
\]
\[
\sum_{i=0}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
\[
\sum_{i=0}^{n} i^k \approx \frac{1}{k+1} n^{k+1}
\]

Infinite Series I

Though we will mostly deal with finite series (i.e. an upper limit of \(n\) for a fixed integer), infinite series are also useful.

Example

Infinite Series II

Consider the geometric series
\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots
\]
This series converges to 2. However, the geometric series
\[
\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \cdots
\]
does not converge. However, note that \(\sum_{n=0}^{\infty} 2^n = 2^{n+1} - 1\)

In fact, we can generalize this as follows.

Lemma

Infinite Series III

A geometric series converges if and only if the absolute value of the common ratio is less than 1.